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FORCE AND TORQUE FORMULAE FOR A SPHERE MOVING IN AN AXISYMMETRIC STOKES FLOW WITH FINITE BOUNDARIES: ASYMMETRIC STOKESLETS NEAR A HOLE IN A PLANE WALL

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Abstract—A sphere is allowed to move with three degrees of freedom in an axisymmetric flow field and general formulae, correct to the third power of the sphere's radius, are developed for the Stokes resistance experienced by the sphere. These are shown to depend on the behaviour within the sphere of the reflected velocity fields which arise from the presence of fixed boundaries at finite distances from stokeslets placed at the sphere's center. Application is made to the stagnation flow at a plane, Poiseuille flow and flow past a sphere and some comparisons made with exact formulae. Solutions are given for asymmetrically placed stokeslets near a hole in a plane wall or a disk.

1. INTRODUCTION

Many problems in chemical engineering concern the motion of a viscous fluid containing suspended particles in the presence of fixed boundaries. When, as is often the case, the viscous stresses dominate the inertia effects, the fluid flow is essentially described by the creeping flow equations but this zero Reynolds number simplification has the disadvantage of slower convergence of numerical computations because the hydrodynamic interaction between a particle and a fixed boundary decays in 3-D only at the rate of inverse distance. This same disadvantage also applies to the commonly used mathematical device known as the method of reflections in which a solution is constructed iteratively by alternately ignoring the boundaries and the particles. The restrictions placed upon particle size and position by this weak interaction technique are eliminated by the recently expanded strong interaction theories available from mathematical analysis applicable when it is possible to fit all boundaries into a single family of coordinate surfaces. Thus for exact analyses the geometry is restricted to two spheres or one sphere and a plane. Hence, for other geometries, Ganatos *et al.* (1978) have developed a strong interaction theory in which a numerical collocation technique is applied to exactly formulated solutions of the equations of motion. It is amply illustrated by the calculations made by these authors (1980) for a sphere between parallel planes.

There remains though the motivation to widen, alongside this computational progress, the scope of mathematical analysis by developing a uniform approximation procedure for a single sphere moving with the appropriate three degrees of freedom in an axisymmetric flow. First order corrections in terms of the sphere radius have been listed by Happel & Brenner (1973) for the force and torque coefficients and the second order contributions arise from the well known Faxén laws. In this paper general formulae, correct to third order, are developed by an essentially simple, though algebraically complicated, application of the method of matched asymptotic expansions which exploits the fact that near the sphere the fluid is almost unaware of the fixed axisymmetric boundaries whilst away from the sphere the latter appears as a point singularity. The "near field" solution is equivalent to the Lamb's spherical harmonic expansions employed by Ganatos *et al.* (1978, 1980) but in the "far field", the additional velocities are due to appropriate singularities—

stokeslet, dipole etc.—placed at the center of the sphere and their calculation depends only on the geometry of each directed singularity with the axisymmetric fixed boundaries. The required strengths of these singularities are determined up to the third power of the sphere radius ϵ by the condition that they be the same in the two constructed solutions. In contrast to the method of reflections, only terms which can contribute to the required order of approximation are retained in the calculation. The meridianal angle is measured, without loss of generality, from the sphere's center and the formulae obtained for the two force components and torque involve values and derivatives of the imposed fluid flow and the singular fields at the sphere center only. General experience with the application of asymptotic methods suggests that the assumption that the sphere is small is unlikely to be as restrictive as might be expected from the mathematical argument. Further, the extension of the method to more than one sphere would appear to be plausible.

Application is made to three disparate cases—stagnation flow at a plane, Poiseuille flow and flow past a sphere—in each of which the pair of force formulae are found to separate. Theorems are given to show that the dipole fields can be obtained from suitable second order derivatives of the corresponding stokeslet fields and to verify that for each of the above cases the stokeslet fields are such that the reflected velocity component parallel to the stokeslet is a symmetric function of the field and singularity positions. Comparison is made with the strong interaction theories available for a sphere moving in a quiescent fluid near a plane and for axisymmetric flow past two relatively moving spheres. A further interesting application is to the pressure driven flow through a circular hole in an infinitesimally thin plane wall, for which the axisymmetric stokeslet field was constructed by Davis *et al.* (1981). Their solution is used to obtain numerical estimates of the force coefficients for an axially placed sphere that show remarkably good agreement with those calculated by Dagan *et al.* (1982) using the above mentioned boundary collocation technique. When the sphere is off-axis, the force formulae do not simplify and there follows the construction of the reflected fields due to asymmetrically placed stokeslets. The normal velocity component at the wall is cancelled by an obvious extension of the axisymmetric solution. However the cancellation of the tangential velocity at the wall requires a field whose every Fourier mode except the zeroth leads to two connected sets of dual integral equations. These are solved by reduction to a single Fredholm integral equation whose solution can be written down by inspection. The relatively minor modifications for the complementary disk problem are included for completeness.

2. FORMULATION OF THE PROBLEM

The equations of motion for a Stokes flow are

$$\operatorname{grad} p = \mu \nabla^2 \mathbf{q}, \qquad \operatorname{div} \mathbf{q} = 0 \qquad [2.1]$$

where $\mathbf{q}(x, y, z)$ is the velocity field, $p(x, y, z)$ the fluid pressure and μ the viscosity. In addition $\mathbf{q}$ must satisfy the no-slip condition at any solid boundaries.

Let the cylindrical polar coordinates (ρ, ω, z) be related to the above Cartesians by $x = \rho \cos \omega$, $y = \rho \sin \omega$ and consider an axisymmetric flow field $\mathbf{W}(\rho, z)$ bounded internally or externally by an axisymmetric boundary S which may or may not intersect the axis $\rho = 0$. Then $\mathbf{W}$ satisfies [2.1] and vanishes on S.

Suppose that this flow field $\mathbf{W}$ is disturbed by the presence of a small solid sphere of radius ϵ, which is moving with velocity $U^*\hat{x} + V^*\hat{z}$ and rotating with angular velocity $\Omega^*\hat{y}$ and whose center is instantaneously at $(x_0, 0, z_0)$. These three components are the only ones which can be induced from rest by the axisymmetric flow. It will be assumed that the Reynolds number of the flow is sufficiently small for inertia effects to be ignored. Then, in

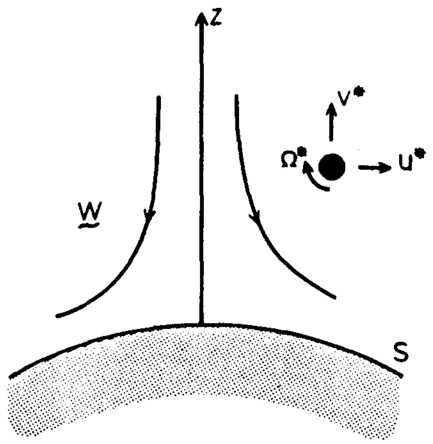

Figure 1. A sketch of envisaged flow.

this quasi-static approximation, the disturbance velocity field $\mathbf{w}(x, y, z)$ satisfies [2.1] and the boundary conditions

$$\mathbf{w} = 0 \quad \text{on} \quad S \qquad [2.2]$$

$$\mathbf{w} \to 0 \quad \text{as} \quad (\rho^2 + z^2)^{1/2} \to \infty \qquad [2.3]$$

$$\mathbf{q} = \mathbf{W} + \mathbf{w} = U^*\hat{x} + V^*\hat{z} + \Omega^*[(z - z_0)\hat{x} - (x - x_0)\hat{z}] \quad \text{at} \quad r = \epsilon \qquad [2.4]$$

where $r^2 = (x - x_0)^2 + y^2 + (z - z_0)^2$.

The required quantities here are the force $(F_x\hat{x} + F_z\hat{z})$ and torque $L\hat{y}$ exerted by the fluid motion on the small moving sphere and will be calculated up to terms of order ϵ^4 by a method based on inner and outer expansions. For small values of ϵ, the fluid near the sphere is essentially unaware of the fixed boundary whilst to the fluid far from the sphere, it appears to be a point singularity. The use of inner and outer coordinates can be avoided since all expansions are regular in ϵ.

3. DERIVATION OF THE FORCE AND TORQUE FORMULAE

The imposed velocity field $\mathbf{W} = W_\rho(\rho, z)\hat{\rho} + W_z(\rho, z)\hat{z}$ has Cartesian components $W_x = (x/\rho)W_\rho$, $W_y = (y/\rho)W_\rho$ and W_z and for the purpose of satisfying condition [2.4], a suitable expansion of $\mathbf{W}$ in the neighbourhood of $(x_0, 0, z_0)$ is required. The Taylor expansion is simplified by the divergence free property of $\mathbf{W}$ and by the even, odd and even dependence on y of W_x, W_y and W_z respectively. However, since [2.4] is to be applied at $r = \epsilon$, a suitable representation of $\mathbf{W}$ in the "inner field" is

$$\mathbf{W} = \text{curl}\,[X(r, \theta, \phi)\hat{r}] + \text{curl}^2\,[\Psi(r, \theta, \phi)\hat{r}] \qquad [3.1]$$

where

$$\nabla^2(r^{-1}X) = 0 \qquad [3.2a]$$

$$\nabla^4(r^{-1}\Psi) = 0\,. \qquad [3.2b]$$

Because the above mentioned Taylor expansion is symmetric in the pairs (x, x_0) and (z, z_0), it is convenient that the spherical polar angles in [3.1] be defined so that the axis is in the $\hat{y}$ direction. Thus write

$$x - x_0 = r\sin\theta\sin\phi, \qquad y = r\cos\theta, \qquad z = r\sin\theta\cos\phi\,.$$

Then, when the Taylor expansion of W up to second order is written in the form (3.1), it follows, with P_n^m denoting an associated Legendre function, that:

$$X \sim \frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) r^2 P_1(\cos\theta) + \frac{1}{18} r^3 P_2^1(\cos\theta)(a_1\cos\phi - b_1\sin\phi) \qquad [3.3]$$

$$\Psi \sim \frac{1}{2} r^2 P_1^1(\cos\theta)(W_x\sin\phi + W_z\cos\phi) + \frac{1}{6}\frac{\partial W_y}{\partial y} r^3 P_2(\cos\theta)$$

$$+ \frac{1}{36} r^3 P_2^2(\cos\theta)\left[-\left(\frac{\partial W_x}{\partial x} - \frac{\partial W_z}{\partial z}\right)\cos 2\phi + \left(\frac{\partial W_x}{\partial z} + \frac{\partial W_z}{\partial x}\right)\sin 2\phi\right]$$

$$+ \frac{1}{36} r^4 P_3^1(\cos\theta)(a_2\sin\phi + b_2\cos\phi) + r^4 P_1^1(\cos\theta)(a_3\sin\phi + b_3\cos\phi)$$

$$+ \frac{1}{720} r^4 P_3^3(\cos\theta)(a_4\sin 3\phi + b_4\cos 3\phi). \qquad [3.4]$$

Here the disjoint sets of coefficients $\{a_j; 1 \le j \le 4\}$ and $\{b_j; 1 \le j \le 4\}$ are given by

$$a_1 = -\left(\frac{\partial^2 W_z}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial x\partial y}\right) + \left(\frac{\partial^2 W_x}{\partial z^2} - \frac{\partial^2 W_x}{\partial y^2}\right), \quad b_1 = -\left(\frac{\partial^2 W_x}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial y\partial z}\right) + \left(\frac{\partial^2 W_z}{\partial x^2} - \frac{\partial^2 W_z}{\partial y^2}\right)$$

$$a_2 = -\frac{1}{4}\left(\frac{\partial^2 W_z}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial x\partial y}\right) - \frac{1}{4}\left(\frac{\partial^2 W_x}{\partial x^2} - \frac{\partial^2 W_x}{\partial y^2}\right) - \frac{1}{20}\nabla^2 W_x, \qquad a_3 = \frac{1}{20}\nabla^2 W_x$$

$$b_2 = -\frac{1}{4}\left(\frac{\partial^2 W_x}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial y\partial z}\right) - \frac{1}{4}\left(\frac{\partial^2 W_z}{\partial z^2} - \frac{\partial^2 W_z}{\partial y^2}\right) - \frac{1}{20}\nabla^2 W_z, \qquad b_3 = \frac{1}{20}\nabla^2 W_z \qquad [3.5]$$

$$a_4 = \frac{1}{2}\left(\frac{\partial^2 W_z}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial x\partial y}\right) - \frac{1}{6}\nabla^2 W_x + \frac{2}{3}\left(\frac{\partial^2 W_x}{\partial z^2} - \frac{\partial^2 W_x}{\partial y^2}\right) - \frac{5}{6}\left(\frac{\partial^2 W_x}{\partial x^2} - \frac{\partial^2 W_x}{\partial y^2}\right)$$

$$b_4 = \frac{1}{2}\left(\frac{\partial^2 W_x}{\partial x\partial z} - \frac{\partial^2 W_y}{\partial y\partial z}\right) - \frac{1}{6}\nabla^2 W_z + \frac{2}{3}\left(\frac{\partial^2 W_z}{\partial x^2} - \frac{\partial^2 W_z}{\partial y^2}\right) - \frac{5}{6}\left(\frac{\partial^2 W_z}{\partial z^2} - \frac{\partial^2 W_z}{\partial y^2}\right)$$

and the components of W and their derivatives are evaluated at $(x_0, 0, z_0)$.

Now if the additional velocity field w is represented in the form

$$\mathbf{w} = \text{curl}\,\{\chi(r, \theta, \phi)\hat{r}\} + \text{curl}^2\,[\psi(r, \theta, \phi)\hat{r}] \qquad [3.6]$$

where χ, ψ satisfy [3,2a, b] respectively, then [2.4] implies that χ and ψ must satisfy the boundary conditions

$$\left.\begin{aligned} \chi + X &= \Omega^* r^2\cos\theta \\ \psi + \Psi &= \frac{1}{2} r^2 \sin\theta(U^*\sin\phi + V^*\cos\phi) \\ \frac{\partial}{\partial r}(\psi + \Psi) &= r\sin\theta(U^*\sin\phi + V^*\cos\phi) \end{aligned}\right\} \text{ at } \quad r = \epsilon. \qquad [3.7]$$

The other boundary conditions [2.2] and [2.3] on w apply to the "outer field" where the representation [3.6] is inappropriate.

Expansions [3.3] and [3.4] show that the reflection of the imposed velocity field **W** at the sphere boundary $r = \epsilon$ can be obtained by constructing the sets of fundamental velocity fields curl $[\chi_{m,n}(r,\theta)\hat{r}{}^{\cos}_{\sin} m\phi]$, curl^2 $[\psi^{(i)}_{m,n}(r,\theta)\hat{r}{}^{\cos}_{\sin} m\phi]$ $(i = 0, 1)$ such that, for $0 \leq m \leq n$ and $n \geqslant 1$,

$$\left.\begin{matrix}\chi_{m,n}\\ \psi^{(0)}_{m,n}\end{matrix}\right\} \sim r^{n+1}P_n^m(\cos\theta), \qquad \psi^{(1)}_{m,n} \sim r^{n+3}P_n^m(\cos\theta) \quad \text{as} \quad r\to\infty$$

$[\chi_{m,n}, \psi^{(i)}_{m,n}$ satisfy [3.2a, b] respectively]

$$\chi_{m,n} = 0 = \psi^{(i)}_{m,n} = \frac{\partial}{\partial r}\psi^{(i)}_{m,n} \quad \text{at} \quad r = \epsilon.$$

Then, by elementary calculation,

$$\chi_{m,n} = \left(r^{n+1} - \frac{\epsilon^{2n+1}}{r^n}\right)P_n^m(\cos\theta) \qquad [3.8]$$

$$\left.\begin{aligned}\psi^{(0)}_{m,n} &= \left[r^{n+1} - \left(n+\frac{1}{2}\right)\frac{\epsilon^{2n-1}}{r^{n-2}} + \left(n-\frac{1}{2}\right)\frac{\epsilon^{2n+1}}{r^n}\right]P_n^m(\cos\theta)\\ \psi^{(1)}_{m,n} &= \left[r^{n+3} - \left(n+\frac{3}{2}\right)\frac{\epsilon^{2n+1}}{r^{n-2}} + \left(n+\frac{1}{2}\right)\frac{\epsilon^{2n+3}}{r^n}\right]P_n^m(\cos\theta)\end{aligned}\right\} \quad (n \geqslant 1) \qquad [3.9]$$

In [3.9] the restriction $n \geqslant 1$ ensures that all reflected velocities vanish as $r \to \infty$. Corresponding to the solutions [3.9], the net force exerted on the sphere by the fluid is again zero for $n \geqslant 2$ and when $n = 1$ is $12\pi\mu\epsilon(\hat{z}, \hat{x}, \hat{y})$ for $\psi^{(0)}_{1,1}\cos\phi$, $\psi^{(0)}_{1,1}\sin\phi$, $\psi^{(0)}_{0,1}$ respectively and $20\pi\mu\epsilon^3(\hat{z}, \hat{x}, \hat{y})$ for $\psi^{(1)}_{1,1}\cos\phi$, $\psi^{(1)}_{1,1}\sin\phi$, $\psi^{(1)}_{0,1}$ respectively. Effectively there is a force $-8\pi\mu(\hat{z}, \hat{x}, \hat{y})$ due to each stokeslet term $(z - z_0, x - x_0, y)$ respectively in ψ but none otherwise. Similarly the solutions [3.8] yield a net torque exerted on the sphere by the fluid which is zero for $n \geqslant 2$ and when $n = 1$ is $8\pi\mu\epsilon^3(\hat{z}, \hat{x}, \hat{y})$ for $\chi_{1,1}\cos\phi$, $\chi_{1,1}\sin\phi$, $\chi_{0,1}$ respectively. Thus there is a torque $-8\pi\mu(\hat{z}, \hat{x}, \hat{y})$ due to each rotlet term $(z - z_0, x - x_0, y)$ respectively in χ but none otherwise.

The contribution to **w** due to the reflection of the field **W** by the sphere without taking account of the fixed boundary S can now be written down by comparison of [3.8] and [3.9] with [3.3], [3.4] and [3.7]. Thus

$$\chi + X - \Omega^* r^2\cos\theta = \chi_{0,1}\left[\frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) - \Omega^*\right] + \frac{1}{18}\chi_{1,2}(a_1\cos\phi - b_1\sin\phi)$$
$$+ \text{further terms} \qquad [3.10a]$$

$$\psi + \Psi - \frac{1}{2}r^2\sin\theta(U^*\sin\phi + V^*\cos\phi) = \frac{1}{2}\psi^{(0)}_{1,1}[(W_x - U^*)\sin\phi + (W_z - V^*)\cos\phi]$$
$$+ \frac{1}{6}\psi^{(0)}_{0,2}\frac{\partial W_y}{\partial y} + \frac{1}{36}\psi^{(0)}_{2,2}\left[\left(\frac{\partial W_x}{\partial z} + \frac{\partial W_z}{\partial x}\right)\sin 2\phi - \left(\frac{\partial W_x}{\partial x} - \frac{\partial W_z}{\partial z}\right)\cos 2\phi\right]$$
$$+ \frac{1}{36}\psi^{(1)}_{1,3}(a_2\sin\phi + b_2\cos\phi) + \psi^{(1)}_{1,1}(a_3\sin\phi + b_3\cos\phi)$$
$$+ \frac{1}{720}\psi^{(0)}_{3,3}(a_4\sin 3\phi + b_4\cos 3\phi) + \text{further terms}. \qquad [3.10b]$$

Consideration of the $n = 1$ terms in [3.10a, b] shows that, at this stage of the calculation, the torque is in the $\hat{y}$ direction and normal to the force, in agreement with the axial symmetry of **W**. Further, of the eight coefficients given by [3.5], only the Laplacian pair a_3 and b_3 can enter the required formulae for F_x and F_z whilst none affects the torque coefficient L. Evidently the neglect of higher order terms in the expansions [3.3] and [3.4] causes no errors in the F_x, F_z and L formulae since powers of $r \geqslant 4$ in [3.3] and $\geqslant 5$ in [3.4] yield values of $n \geqslant 2$ in [3.10a, b].

The further terms in [3.10a, b] involve positive powers of ϵ and are due to reflections from the rigid boundary S. Since only terms up to order ϵ^4 in the "far field" are sought, it follows from [3.8] and [3.9] that $\chi_{1,2}$, $\psi^{(1)}_{1,3}$ and $\psi^{(0)}_{3,3}$ may be discarded from [3.10a, b]. Then the remaining functions are given, to the required accuracy, by

$$\left.\begin{aligned} \chi_{0,1} &= \left(r^2 - \frac{\epsilon^3}{r}\right)\cos\theta, \\ \psi^{(0)}_{1,1} &= \left(r^2 - \frac{3}{2}\epsilon r + \frac{\epsilon^3}{2r}\right)\sin\theta, \quad \psi^{(0)}_{0,2} = \left[r^3 - \frac{5}{2}\epsilon^3 + O(\epsilon^5)\right]P_2(\cos\theta) \\ \psi^{(1)}_{1,1} &= \left[r^4 - \frac{5}{2}\epsilon^3 r + O(\epsilon^5)\right]\sin\theta, \quad \psi^{(0)}_{2,2} = \left[r^3 - \frac{5}{2}\epsilon^3 + O(\epsilon^5)\right]P_2^2(\cos\theta) \end{aligned}\right\} \quad [3.11]$$

where $P_2^2(\cos\theta) = 3\sin^2\theta$. So the "far field" behaviour of [3.10a, b] contains, to order ϵ^4, terms which correspond to the following velocity singularities at $r = 0$. From $\chi_{0,1}$ there is

$$\operatorname{curl}\left(\frac{\cos\theta}{r}\hat{r}\right) = \frac{\sin\theta}{r^2}\hat{\phi} = \frac{1}{r^3}[(z - z_0)\hat{x} - (x - x_0)\hat{z}] \quad [3.12a]$$

which is a rotlet in the $\hat{y}$ direction. From $\psi^{(0)}_{1,1}$ and $\psi^{(1)}_{1,1}$, there are

$$\operatorname{curl}^2\left(r\sin\theta\,\hat{r}\ \begin{matrix}\sin\\ \cos\end{matrix}\ \phi\right) = \frac{\hat{r}}{r^2}\begin{pmatrix}x - x_0\\ z - z_0\end{pmatrix} + \frac{1}{r}\begin{pmatrix}\hat{x}\\ \hat{z}\end{pmatrix} \quad [3.12b]$$

which are stokeslets in the $\hat{x}$ and $\hat{z}$ directions respectively. From $\psi^{(0)}_{1,1}$, there are

$$\operatorname{curl}^2\left(\frac{\sin\theta}{r}\hat{r}\ \begin{matrix}\sin\\ \cos\end{matrix}\ \phi\right) = \frac{3\hat{r}}{r^4}\begin{pmatrix}x - x_0\\ z - z_0\end{pmatrix} - \frac{1}{r^3}\begin{pmatrix}\hat{x}\\ \hat{z}\end{pmatrix} \quad [3.12c]$$

which are dipoles in the $\hat{x}$ and $\hat{z}$ directions respectively. From $\psi^{(0)}_{0,2}$ and $\psi^{(0)}_{2,2}$, there are

$$\operatorname{curl}^2\begin{bmatrix} -\frac{1}{3}P_2(\cos\theta)\hat{r} \\ \frac{1}{6}P_2^2(\cos\theta)\hat{r}\ \begin{matrix}\cos\\ \sin\end{matrix}\ 2\phi \end{bmatrix} = \begin{bmatrix}\partial/\partial x_0\\ -\partial/\partial x_0\\ \partial/\partial z_0\end{bmatrix}\left[\frac{\hat{r}}{r^2}(x - x_0) + \frac{\hat{x}}{r}\right] + \begin{bmatrix}\partial/\partial z_0\\ \partial/\partial z_0\\ \partial/\partial x_0\end{bmatrix}\left[\frac{\hat{r}}{r^2}(z - z_0) + \frac{\hat{z}}{r}\right]$$

[3.13]

which are combinations of the x_0 and z_0 derivatives of the stokeslet singularities. The remaining combination yields the rotlet singularity [3.12a].

The additional terms in the expressions [3.10a, b] for χ and ψ arise firstly from the reflections by S of the above singular velocity fields. Let **Ω**, **U**, **V**, **u** and **v** be velocity fields which satisfy [2.1], [2.3] and are such that

$$\left.\begin{aligned}&\mathbf{\Omega}=\frac{1}{r^3}[(z-z_0)\hat{x}-(x-x_0)\hat{z}],\\&\mathbf{U}=\frac{\hat{r}}{r^2}(x-x_0)+\frac{\hat{x}}{r},\quad \mathbf{V}=\frac{\hat{r}}{r^2}(z-z_0)+\frac{\hat{z}}{r}\\&\mathbf{u}=\frac{3\hat{r}}{r^4}(x-x_0)-\frac{\hat{x}}{r^3},\quad \mathbf{v}=\frac{3\hat{r}}{r^4}(z-z_0)-\frac{\hat{z}}{r^3}\end{aligned}\right\}\text{ on } S. \qquad [3.14]$$

Then the required singular velocity fields, satisfying [2.1], [2.2] and [2.3], are

$$\frac{1}{r^3}[(z-z_0)\hat{x}-(x-x_0)\hat{z}]-\mathbf{\Omega}(x,y,z;x_0,z_0) \qquad \text{(rotlet)}$$

$$\left.\begin{aligned}&\frac{\hat{r}}{r^2}(x-x_0)+\frac{\hat{x}}{r}-\mathbf{U}(x,y,z;x_0,z_0)\\&\frac{\hat{r}}{r^2}(z-z_0)+\frac{\hat{z}}{r}-\mathbf{V}(x,y,z;x_0,z_0)\end{aligned}\right\} \qquad \text{(stokeslets)}$$

$$\left.\begin{aligned}&\frac{3\hat{r}}{r^4}(x-x_0)-\frac{\hat{x}}{r^3}-\mathbf{u}(x,y,z;x_0,z_0)\\&\frac{3\hat{r}}{r^4}(z-z_0)-\frac{\hat{z}}{r^3}-\mathbf{v}(x,y,z;x_0,z_0)\end{aligned}\right\} \qquad \text{(dipoles)}$$

and, from [3.13]

$$\left(\frac{1}{r^2}-\frac{3y^2}{r^4}\right)\hat{r}-\partial\mathbf{U}/\partial x_0-\partial\mathbf{V}/\partial z_0$$

$$\frac{3\hat{r}}{r^4}[(z-z_0)^2-(x-x_0)^2]+\partial\mathbf{U}/\partial x_0-\partial\mathbf{V}/\partial z_0$$

$$\frac{6\hat{r}}{r^4}(x-x_0)(z-z_0)-\partial\mathbf{U}/\partial z_0-\partial\mathbf{V}/\partial x_0.$$

The remaining combination of derivatives yields an expression for $\mathbf{\Omega}$ in terms of $\mathbf{U}$ and $\mathbf{V}$, namely

$$\mathbf{\Omega}=\frac{1}{2}\left(\frac{\partial\mathbf{U}}{\partial z_0}-\frac{\partial\mathbf{V}}{\partial x_0}\right). \qquad [3.15]$$

The expansions near $r=0$ of these reflected velocity fields are required to second or zero order in r according as the corresponding singularity in [3.11] is of order ϵ or ϵ^3 respectively and are of the same form as that given for $\mathbf{W}$ by [3.1], [3.3], [3.4] and [3.5]. Only the first term of [3.4] contributes to the zero order terms of $\mathbf{W}$ and the required expansions of $\partial\mathbf{U}/\partial x_0$ etc. can also be written in the simpler forms

$$\frac{\partial\mathbf{U}}{\partial x_0}\sim\operatorname{curl}^2\left\{\frac{1}{2}r^2\sin\theta\left[\frac{\partial U_x}{\partial x_0}\sin\phi+\frac{\partial U_z}{\partial x_0}\cos\phi\right]\hat{r}\right\},\text{ etc.,}$$

provided it is understood that x_0 and z_0 derivatives are taken before (x, y, z) is set equal to $(x_0, 0, z_0)$. Since, for a differentiable function f,

$$\left[\frac{\partial}{\partial x_0} f(x, x_0)\right]_{x=x_0} = \frac{\partial}{\partial x_0} f(x_0, x_0) - \left[\frac{\partial}{\partial x} f(x, x_0)\right]_{x=x_0} \quad [3.16]$$

the consistency with the result obtained by differentiating the Taylor series of U is evident.

Thus the disturbance velocity field **w** has "outer field" expansion of the form

$$\begin{aligned}\mathbf{W} \sim\ & A(\epsilon)[\mathrm{curl}^2 (r \sin\theta\, \hat{r} \sin\phi) - \mathbf{U}] + B(\epsilon)[\mathrm{curl}^2 (r \sin\theta\, \hat{r} \cos\phi) - \mathbf{V}] \\ & + C_1(\epsilon)\left[\mathrm{curl}^2\left(\frac{1}{r}\sin\theta\, \hat{r} \sin\phi\right) - \mathbf{u}\right] + C_2(\epsilon)\left[\mathrm{curl}^2\left(\frac{1}{r}\sin\theta\, \hat{r}\cos\phi\right) - \mathbf{v}\right] \\ & + D_1(\epsilon)\left[\mathrm{curl}^2\left(-\frac{1}{3} P_2(\cos\theta)\, \hat{r}\right) - \frac{\partial \mathbf{U}}{\partial x_0} - \frac{\partial \mathbf{V}}{\partial z_0}\right] \\ & + D_2(\epsilon)\left[\mathrm{curl}^2\left(\frac{1}{6} P_2^2(\cos\theta)\, \hat{r}\cos 2\phi\right) + \frac{\partial \mathbf{U}}{\partial x_0} - \frac{\partial \mathbf{V}}{\partial z_0}\right] \\ & + D_3(\epsilon)\left[\mathrm{curl}^2\left(\frac{1}{6} P_2^2(\cos\theta)\, \hat{r}\sin 2\phi\right) - \frac{\partial \mathbf{U}}{\partial z_0} - \frac{\partial \mathbf{V}}{\partial x_0}\right] \\ & + E(\epsilon)\left[\mathrm{curl}\left(\frac{1}{r}\cos\theta\, \hat{r}\right) - \mathbf{\Omega}\right] \end{aligned} \quad [3.17]$$

where, as observed earlier, [3.10a, b] and [3.11] imply that $A(\epsilon)$, $B(\epsilon) = O(\epsilon)$ whilst the remaining coefficients are $O(\epsilon^3)$. Then the reflected velocities due to S are themselves reflected by the sphere in the same way as the prescribed flow **W** and so the required further terms in [3.10a, b] may be written down by comparison of the singular field expansions with [3.3], [3.4] and [3.5]. Thus

$$\begin{aligned}\chi + X - \Omega^* r^2 \cos\theta = \chi_{0,1}\Bigg[& \frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) - \Omega^* - \frac{1}{2}A(\epsilon)\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) \\ & - \frac{1}{2}B(\epsilon)\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + O(\epsilon^3)\Bigg] \\ & + \text{torque-free terms} \end{aligned} \quad [3.18a]$$

$$\begin{aligned}\psi + \Psi - \frac{1}{2} r^2 \sin\theta (U^* \sin\phi + V^* \cos\phi) = & \frac{1}{2}\psi_{1,1}^{(0)}[(W_x - U^*)\sin\phi + (W_z - V^*)\cos\phi] \\ & + \frac{1}{6}\psi_{0,2}^{(0)}\frac{\partial W_y}{\partial y} + \frac{1}{36}\psi_{2,2}^{(0)}\left[\left(\frac{\partial W_x}{\partial z} + \frac{\partial W_z}{\partial x}\right)\sin 2\phi - \left(\frac{\partial W_x}{\partial x} - \frac{\partial W_z}{\partial z}\right)\cos 2\phi\right] \\ & + \frac{1}{20}\psi_{1,1}^{(1)}(\nabla^2 W_x \sin\phi + \nabla^2 W_z \cos\phi) - A(\epsilon)\left\{\frac{1}{2}\psi_{1,1}^{(0)}(U_x \sin\phi + U_z\cos\phi) + \frac{1}{6}\psi_{0,2}^{(0)}\frac{\partial U_y}{\partial y}\right. \\ & + \frac{1}{36}\psi_{2,2}^{(0)}\left[\left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x}\right)\sin 2\phi - \left(\frac{\partial U_x}{\partial x} - \frac{\partial U_z}{\partial z}\right)\cos 2\phi\right] \\ & \left. + \frac{1}{20}\psi_{1,1}^{(1)}[\nabla^2 U_x \sin\phi + \nabla^2 U_z \cos\phi]\right\} - B(\epsilon)\left\{\frac{1}{2}\psi_{1,1}^{(0)}(V_x \sin\phi + V_z\cos\phi)\right.\end{aligned}$$

$$+\frac{1}{6}\psi^{(0)}_{0,2}\frac{\partial V_y}{\partial y}+\frac{1}{36}\psi^{(0)}_{2,2}\left[\left(\frac{\partial V_x}{\partial z}+\frac{\partial V_z}{\partial x}\right)\sin 2\phi-\left(\frac{\partial V_x}{\partial x}-\frac{\partial V_z}{\partial z}\right)\cos 2\phi\right]$$

$$+\frac{1}{20}\psi^{(1)}_{1,1}[\nabla^2 V_x\sin\phi+\nabla^2 V_z\cos\phi]\bigg\}-\frac{1}{2}\psi^{(0)}_{1,1}\bigg\{C_1(\epsilon)(u_x\sin\phi+u_z\cos\phi)$$

$$+C_2(\epsilon)(v_x\sin\phi+v_z\cos\phi)+E(\epsilon)(\Omega_x\sin\phi+\Omega_z\cos\phi)$$

$$+[D_1(\epsilon)+D_2(\epsilon)]\left[\frac{\partial U_x}{\partial x_0}\sin\phi+\frac{\partial U_z}{\partial x_0}\cos\phi\right]$$

$$+[D_1(\epsilon)-D_2(\epsilon)]\left[\frac{\partial V_x}{\partial z_0}\sin\phi+\frac{\partial V_z}{\partial z_0}\cos\phi\right]$$

$$+D_3(\epsilon)\left[\left(\frac{\partial U_x}{\partial z_0}+\frac{\partial V_x}{\partial x_0}\right)\sin\phi+\left(\frac{\partial U_z}{\partial z_0}+\frac{\partial V_z}{\partial x_0}\right)\cos\phi\right]\bigg\}$$

$$+O(\epsilon^5)+\text{force-free terms.} \qquad [3.18b]$$

Equations for $A(\epsilon)$ etc. can now be obtained by using [3.8] and [3.9] to equate the strengths of the respective singularities at $r=0$ in each of the velocity fields given by [3.17] and [3.18a, b]. The force and torque exerted by the fluid on the sphere are given, from the stokeslet and rotlet singularities, by

$$F_x\hat{x}+F_z\hat{z}=-8\pi\mu[A(\epsilon)\hat{x}+B(\epsilon)\hat{z}]$$

$$L\hat{y}=-8\pi\mu E(\epsilon)\hat{y}.$$

The effective Stokes relative velocity components $(6\pi\mu\epsilon)^{-1}(F_x, F_z)=-(4/3\epsilon)[A(\epsilon), B(\epsilon)]$ are then found, after eliminating the other coefficients from the above mentioned equations, without inverting any expansions, to be given to order ϵ^3 by the simultaneous equations

$$\frac{F_x}{6\pi\mu\epsilon}\left\{1-\frac{3}{4}\epsilon U_x+\frac{1}{4}\epsilon^3 u_x-\frac{1}{8}\epsilon^3\nabla^2 U_x\right\}+\frac{F_z}{6\pi\mu\epsilon}\left\{-\frac{3}{4}\epsilon V_x+\frac{1}{4}\epsilon^3 v_x-\frac{1}{8}\epsilon^3\nabla^2 V_x\right\}$$

$$-W_x-U^*+\frac{1}{6}\epsilon^2\nabla^2 W_x+\epsilon^3\left[\frac{1}{2}\left(\frac{\partial W_x}{\partial z}-\frac{\partial W_z}{\partial x}\right)-\Omega^*\right]\Omega_x-\frac{5}{4}\epsilon^3 Q_x \qquad [3.19a]$$

$$\frac{F_x}{6\pi\mu\epsilon}\left\{-\frac{3}{4}\epsilon U_z+\frac{1}{4}\epsilon^3 u_z-\frac{1}{8}\epsilon^3\nabla^2 U_z\right\}+\frac{F_z}{6\pi\mu\epsilon}\left\{1-\frac{3}{4}\epsilon V_z+\frac{1}{4}\epsilon^3 v_z-\frac{1}{8}\epsilon^3\nabla^2 V_z\right\}$$

$$\sim W_z-V^*+\frac{1}{6}\epsilon^2\nabla^2 W_z+\epsilon^3\left[\frac{1}{2}\left(\frac{\partial W_x}{\partial z}-\frac{\partial W_z}{\partial x}\right)-\Omega^*\right]\Omega_z-\frac{5}{4}\epsilon^3 Q_z \qquad [3.19b]$$

where the coefficients of F_x and F_z depend only on the location of S relative to the point $(x_0, 0, z_0)$ and the velocity components Q_x, Q_z are given by

$$(Q_x, Q_z)=\left[\frac{\partial W_y}{\partial y}-\frac{1}{3}\left(\frac{\partial W_x}{\partial x}-\frac{\partial W_z}{\partial z}\right)\right]\frac{\partial}{\partial x_0}(U_x, U_z)+\left[\frac{\partial W_y}{\partial y}+\frac{1}{3}\left(\frac{\partial W_x}{\partial x}-\frac{\partial W_z}{\partial z}\right)\right]\frac{\partial}{\partial z_0}(V_x, V_z)$$

$$-\frac{1}{3}\left(\frac{\partial W_x}{\partial z}+\frac{\partial W_z}{\partial x}\right)\left[\frac{\partial}{\partial z_0}(U_x, U_z)+\frac{\partial}{\partial x_0}(V_x, V_z)\right] \qquad [3.20]$$

Also the effective relative angular velocity is given by

$$\frac{L}{8\pi\mu\epsilon^3} =$$

$$-\frac{E(\epsilon)}{\epsilon^3} = \frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) - \Omega^* + \frac{F_x}{16\pi\mu}\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) + \frac{F_z}{16\pi\mu}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + O(\epsilon^3)$$

$$= \frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) - \Omega^* + \frac{3}{8}\epsilon\left\{\left(1 + \frac{3}{4}\epsilon U_x\right)(W_x - U^*) + \frac{3}{4}\epsilon V_x(W_z - V^*)\right\}\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right)$$

$$+ \frac{3}{8}\epsilon\left\{\frac{3}{4}\epsilon U_z(W_x - U^*) + \left(1 + \frac{3}{4}\epsilon V_z\right)(W_z - V^*)\right\}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + O(\epsilon^3) \quad [3.21]$$

after inserting the $O(\epsilon)$ solution of [3.19a, b]. Here the $O(1)$ term is that predicted by Faxen's law.

For the neutrally buoyant sphere, U^*, V^* and Ω^* are chosen so that the force components and torque are all zero. Then, from [3.19a, b] and [3.21],

$$\left.\begin{aligned} U^* &\sim W_x + \frac{1}{6}\epsilon^2\nabla^2 W_x - \frac{5}{4}\epsilon^3 Q_x \\ V^* &\sim W_z + \frac{1}{6}\epsilon^2\nabla^2 W_z - \frac{5}{4}\epsilon^3 Q_z \\ \Omega^* &\sim \frac{1}{2}\left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right) + O(\epsilon^3) \end{aligned}\right\} . \quad [3.22]$$

Thus the velocity components Q_x and Q_z, which depend on the imposed velocity field **W** and the location of S relative to $(x_0, 0, z_0)$, indicate the accuracy of the estimates of U^* and V^* obtained by the use of Faxen's law.

For a freely moving sphere whose density is Δ more than that of the fluid, the buoyancy force $-\frac{4}{3}\pi\epsilon^3\Delta g\hat{z}$ must be balanced. Then, on inserting $F_x = 0$, $(F_z/6\pi\mu\epsilon) = 2\epsilon^2\Delta g/9\mu$ and $L = 0$ in [3.19a, b] and [3.21] respectively, the first two of equations [3.22] are replaced by

$$U^* \sim W_x + \frac{1}{6}\epsilon^2\nabla^2 W_x - \frac{5}{4}\epsilon^3 Q_x + \epsilon^3 V_x \Delta g/6\mu$$

$$V^* \sim W_z + \frac{1}{6}\epsilon^2\nabla^2 W_z - \frac{5}{4}\epsilon^3 Q_z - \epsilon^2\left(\frac{4}{3} - \epsilon V_z\right)\Delta g/6\mu$$

whilst the last is unchanged in form.

In the axisymmetric case F_x and L vanish whilst [3.19b] reduces, with the aid of [3.20], to

$$\frac{F_z}{6\pi\mu\epsilon} \sim \frac{W_z - V^* + \frac{1}{6}\epsilon^2\nabla^2 W_z + \frac{5}{4}\epsilon^3\frac{\partial W_z}{\partial z}\cdot\frac{\partial V_z}{\partial z_0}}{1 - \frac{3}{4}\epsilon V_z + \frac{1}{4}\epsilon^3 v_z - \frac{1}{8}\epsilon^3\nabla^2 V_z}. \quad [3.23]$$

This particular result could have been obtained more simply by applying the above procedure to the Stokes stream functions of **W** and **w**.

The formulae [3.19a, b] and [3.21] represent the superposition of the force and torque results for a moving sphere in a quiescent fluid ($\mathbf{W} = 0$) and a fixed sphere in a moving fluid ($U^* = V^* = \Omega^* = 0$), which cases have usually been considered separately by previous authors.

The singularities in [3.8] and [3.9] are evidently those which appear in the Lamb's spherical harmonic expansions employed by Ganatos *et al.* (1978, 1980). The difference between the two methods lies in the way in which the coefficients are determined. In the boundary collocation technique a solution satisfying [2.1]–[2.3] exactly is constructed and a finite number of singularity coefficients determined numerically by applying condition [2.4] at a finite number of latitudes and longitudes on the sphere. In this section, the fluid flows incident upon the sphere have been expanded about its center in order to construct two solutions [3.17] and [3.18a, b] for **w**, the former valid away from the sphere where [2.4] can be ignored and the latter valid near the sphere where [2.2] is immaterial. The coefficients of the singularities were subsequently determined up to a chosen power of the sphere radius ϵ by requiring that they be the same in each of [3.17] and [3.18a, b].

4. DERIVATION OF u, v FROM U, V: SYMMETRY OF U_x AND V_z

The calculation of the dipole reflected fields **u** and **v** can often be avoided by means of the following result, obtained by use of standard Cartesian tensor notation. Let $t_{jk}(P, P^{(0)})$ be the jth component of the velocity field at $P(x_1, x_2, x_3)$ due to a stokeslet in the direction of Ox_k at the point $P^{(0)}(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$ and p_k the corresponding pressure field. Then

$$t_{jk} = \frac{1}{r}\delta_{jk} + \frac{1}{r^3}(x_j - x_j^{(0)})(x_k - x_k^{(0)}), \qquad p_k = \frac{2\mu}{r^3}(x_k - x_k^{(0)}) \qquad [4.1]$$

where $r^2 = (x_1 - x_1^{(0)})^2 + (x_2 - x_2^{(0)})^2 + (x_3 - x_3^{(0)})^2$, and, from [2.1],

$$\frac{\partial p_k}{\partial x_j} = \mu\nabla^2 t_{jk} = \mu\frac{\partial^2 t_{jk}}{\partial x_i \partial x_i} \qquad [4.2]$$

Also let $-T_{jk}^{(S)}(P, P^{(0)})$ be the jth component of the reflected velocity field due to the rigid boundary S, i.e. $T_{jk}^{(S)}(P, P^{(0)}) = t_{jk}(P, P^{(0)})$ when P lies on S. Now the jth component of the velocity field due to a dipole at $P^{(0)}$ in the direction of Ox_k is

$$-\frac{\partial}{\partial x_j}\left(\frac{x_k - x_k^{(0)}}{r^3}\right) = -\frac{1}{2\mu}\frac{\partial p_k}{\partial x_j} = -\frac{1}{2}\nabla^2 t_{jk}$$

$$= -\frac{1}{2}\frac{\partial^2 t_{jk}}{\partial x_i^{(0)}\partial x_i^{(0)}} = -\frac{1}{2}\nabla_0^2 t_{jk}$$

after using [4.1] and [4.2]. Hence the reflected velocity field due to the presence of S in this dipole field has jth component $-T_{jk}^{(D)}$ given by

$$T_{jk}^{(D)}(P, P^{(0)}) = -\frac{1}{2}\nabla_0^2 T_{jk}^{(S)}(P, P^{(0)}) \qquad [4.3]$$

where ∇_0^2 is the Laplacian with respect to the position $P^{(0)}$ of the stokeslet. Thus, to evaluate the fields **u** and **v** in terms of **U** and **V**, the stokeslets must be first moved to (x_0, y_0, z_0) and then y_0 set equal to zero after application of the operator ∇_0^2. Hence

$$\mathbf{u}(x, y, z; x_0, z_0) = -\frac{1}{2}\left[\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial z_0^2}\right)\mathbf{U}(x, y, z; x_0, y_0, z_0)\right]_{y_0 = 0} \qquad [4.4]$$

and similarly for $\mathbf{v}(x, y, z; x_0, z_0)$. In terms of polar coordinates, this procedure is much simplified for $\mathbf{v}$ since then, with $x_0 = \rho_0$,

$$\mathbf{v}(\rho \cos \omega, \rho \sin \omega, z; \rho_0, z_0)$$

$$= -\frac{1}{2}\left[\left(\frac{\partial^2}{\partial \rho_0^2} + \frac{1}{\rho_0}\frac{\partial}{\partial \rho_0} + \frac{1}{\rho_0^2}\frac{\partial^2}{\partial \omega_0^2} + \frac{\partial^2}{\partial z_0^2}\right) \mathbf{V}(\rho \cos \omega, \rho \sin \omega, z; \rho_0 \cos \omega_0, \rho \sin \omega_0, z_0)\right]_{\omega_0 = 0}$$

$$= -\frac{1}{2}\left[\left(\frac{\partial^2}{\partial \rho_0^2} + \frac{1}{\rho_0}\frac{\partial}{\partial \rho_0} + \frac{1}{\rho_0^2}\frac{\partial^2}{\partial \omega_0^2} + \frac{\partial^2}{\partial z_0^2}\right) \mathbf{V}(\rho \cos (\omega - \omega_0), \rho \sin (\omega - \omega_0), z; \rho_0, z_0)\right]_{\omega_0 = 0}$$

$$= -\frac{1}{2}\left(\frac{\partial^2}{\partial \rho_0^2} + \frac{1}{\rho_0}\frac{\partial}{\partial \rho_0} + \frac{1}{\rho_0^2}\frac{\partial^2}{\partial \omega^2} + \frac{\partial^2}{\partial z_0^2}\right) \mathbf{V}(\rho \cos \omega, \rho \sin \omega, z; \rho_0, z_0). \qquad [4.5]$$

As the $\hat{\rho}$ and $\hat{x}$ directions do not coincide when $y_0 \neq 0$, there is no corresponding simplification for $\mathbf{u}$.

Substitution of the result [4.3] into the left hand sides of [3.19a, b] shows that the third order terms have the alternative form

$$\frac{1}{4}\epsilon^3 \begin{Bmatrix} u_x \\ u_z \\ v_x \\ v_z \end{Bmatrix} - \frac{1}{8}\epsilon^3 \nabla^2 \begin{Bmatrix} U_x \\ U_z \\ V_x \\ V_z \end{Bmatrix} = -\frac{1}{8}\epsilon^3 (\nabla_0^2 + \nabla^2) \begin{Bmatrix} U_x \\ U_z \\ V_x \\ V_z \end{Bmatrix} \qquad [4.6]$$

where, in all quantities, (x, y, z) is set equal to $(x_0, 0, z_0)$ after all differentiations have been completed. Clearly [4.4] shows that, if either component of $\mathbf{U}$ or $\mathbf{V}$ is a symmetric function of the stokeslet and field positions, then the corresponding two terms above are equal.

The symmetry of U_x and V_z is ensured by the following result, obtained by applying Green's theorem to the fields $(t_{jk} - T_{jk}^{(S)})_{(P,P^{(0)})}$ and $(t_{jk} - T_{jk}^{(S)})_{(P,P^{(1)})}$ and simplifying as in Happel & Brenner (1973), section 3.4:

$$T_{kk}^{(S)}(P^{(1)}, P^{(0)}) - T_{kk}^{(S)}(P^{(0)}, P^{(1)})$$

$$= \frac{1}{8\pi}\int_S \left\{(t_{jk} - T_{jk}^{(S)})_{(P,P^{(0)})}\left[n_j \nabla^2 (t_{jk} - T_{jk}^{(S)})_{(P,P^{(1)})} - \frac{\partial}{\partial n}(t_{jk} - T_{jk}^{(S)})_{(P,P^{(1)})}\right]\right.$$

$$\left. - (t_{jk} - T_{jk}^{(S)})_{(P,P^{(1)})}\left[n_j \nabla^2 (t_{jk} - T_{jk}^{(S)})_{(P,P^{(0)})} - \frac{\partial}{\partial n}(t_{jk} - T_{jk}^{(S)})_{(P,P^{(0)})}\right]\right\} \mathrm{d}S \qquad [4.7]$$

where the normal $\mathbf{n}$ to S is directed into the fluid region. Now the condition that $t_{jk} = T_{jk}^{(S)}$ when $P \in S$ makes the right hand side of [4.7] vanish and then $T_{kk}^{(S)}$, the component of reflected velocity parallel to the stokeslet, is a symmetric function of the field and stokeslet positions. So the velocity components $U_x(x, y, z; x_0, y_0, z_0)$ and $V_z(x, y, z; x_0, y_0, z_0)$ are symmetric functions of (x, y, z) and (x_0, y_0, z_0).

5. STAGNATION FLOW AT A PLANE

Here S is the plane $z = 0$ so the velocity fields $\mathbf{\Omega}$, $\mathbf{U}$, $\mathbf{u}$, $\mathbf{V}$ and $\mathbf{v}$ evidently depend on $(x - x_0)$, $(y - y_0)$, z and z_0 only, with y_0 set equal to zero. The boundary conditions [3.14] imply that Ω_x, U_x, u_x, V_z and v_z are even functions of $(x - x_0)$ whilst Ω_z, U_z, u_z, V_x and v_x are odd functions of $(x - x_0)$. When these properties and the stagnation flow $\mathbf{W} = \Lambda z(-x\hat{x} - y\hat{y} + z\hat{z})$ are substituted into [3. 19a, b], it follows that the equations for F_x and F_z separate to yield

$$\frac{F_x}{6\pi\mu\epsilon}\left[1 - \frac{3}{4}\epsilon U_x + \frac{1}{4}\epsilon^3 u_x - \frac{1}{8}\epsilon^3\nabla^2 U_x\right] \sim -\Lambda x_0 z_0 - U^* - \epsilon^3\Omega^*\Omega_x - \frac{5}{4}\epsilon^3 Q_x \quad [5.1a]$$

$$\frac{F_z}{6\pi\mu\epsilon}\left[1 - \frac{3}{4}\epsilon V_z + \frac{1}{4}\epsilon^3 v_z - \frac{1}{8}\epsilon^3\nabla^2 V_z\right] \sim \Lambda z_0^2 - V^* - \frac{5}{4}\epsilon^3 Q_z \quad [5.1b]$$

where Ω_x is given by [3.15] and

$$Q_x = \frac{1}{3}\Lambda x_0\left(\frac{\partial U_x}{\partial z_0} + \frac{\partial V_x}{\partial x_0}\right), \qquad Q_z = -2\Lambda z_0\frac{\partial V_z}{\partial z_0}. \quad [5.2]$$

Also [3.21] takes the form

$$\frac{L}{8\pi\mu\epsilon^3} = -\frac{1}{2}\Lambda x_0 - \Omega^* + \frac{3}{8}\epsilon\left(1 + \frac{3}{4}\epsilon U_x\right)(-\Lambda x_0 z_0 - U^*)\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) + O(\epsilon^3). \quad [5.3]$$

Now, from Blake (1971), the required stokeslet velocity components are given by

$$\left.\begin{aligned}
V_z(x - x_0, y, z; z_0) &= \frac{2}{[\bar{\rho}^2 + (z + z_0)^2]^{1/2}} - \frac{\bar{\rho}^2}{[\bar{\rho}^2 + (z + z_0)^2]^{3/2}} + \frac{4zz_0}{[\bar{\rho}^2 + (z + z_0)^2]^{3/2}}\\
&\quad - \frac{6\bar{\rho}^2 zz_0}{[\bar{\rho}^2 + (z + z_0)^2]^{5/2}}\\
U_x(x - x_0, y, z; z_0) &= \frac{1}{[\bar{\rho}^2 + (z + z_0)^2]^{1/2}} + \frac{(x - x_0)^2}{[\bar{\rho}^2 + (z + z_0)^2]^{3/2}} + \frac{2zz_0}{[\bar{\rho}^2 + (z + z_0)^2]^{3/2}}\\
&\quad - \frac{6zz_0(x - x_0)^2}{[\bar{\rho}^2 + (z + z_0)^2]^{5/2}}\\
\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right)_{(x - x_0, y, z; z_0)} &= -\frac{2(z - z_0)}{[\bar{\rho}^2 + (z + z_0)^2]^{3/2}} - \frac{6z_0(x - x_0)^2}{[\bar{\rho}^2 + (z + z_0)^2]^{5/2}}
\end{aligned}\right\} \quad [5.4]$$

where $\bar{\rho}^2 = (x - x_0)^2 + y^2$. The symmetry of V_z and U_x in z, z_0 ensures that [4.4] yields $v_z = -\frac{1}{2}\nabla^2 V_z$, $u_x = -\frac{1}{2}\nabla^2 U_x$ in formulae [5.1a, b] respectively whilst the expression for $\hat{y}\cdot\text{curl}\,\mathbf{U}$ shows that the torque formula [5.3] reduces to

$$\frac{L}{8\pi\mu\epsilon^3} = -\frac{1}{2}\Lambda x_0 - \Omega^* + O(\epsilon^3). \quad [5.5]$$

Evidently $U_x = \frac{1}{2}V_z$ when $\bar{\rho} = 0$ and, since $(\partial V_x/\partial x)_{\bar{\rho}=0} = -\frac{1}{2}(\partial V_z/\partial z)_{\bar{\rho}=0}$ by continuity, it follows that

$$\frac{\partial V_x}{\partial x_0} = \frac{1}{2}\frac{\partial V_z}{\partial z} = \frac{\partial U_x}{\partial z} \quad \text{when} \quad \bar{\rho} = 0.$$

Then the symmetry of U_x in z, z_0 implies that Ω_x vanishes in [5.1a] and

$$\frac{\partial U_x}{\partial z_0}+\frac{\partial V_x}{\partial x_0}=\frac{\partial V_z}{\partial z_0}=-\frac{3}{4z_0^2}$$

in [5.2]. Thus, by substitution of [5.4], the force formulae [5.1a, b] reduce to

$$\frac{F_x}{6\pi\mu\epsilon}\sim\frac{-\Lambda x_0z_0-U^*+\dfrac{5}{16}\Lambda x_0\epsilon^3/z_0^2}{1-\dfrac{9}{16}(\epsilon/z_0)+\dfrac{1}{8}(\epsilon/z_0)^3} \qquad [5.6a]$$

$$\frac{F_z}{6\pi\mu\epsilon}\sim\frac{\Lambda z_0^2-V^*+\dfrac{1}{3}\epsilon^2\Lambda-\dfrac{15}{8}\epsilon^3\Lambda/z_0}{1-\dfrac{9}{8}(\epsilon/z_0)+\dfrac{1}{2}(\epsilon/z_0)^3}. \qquad [5.6b]$$

Note that x_0 does not appear in [5.6b]. Comparison of the force formulae [5.6a, b] with their counterparts (Brenner 1962) for the same sphere moving in a quiescent fluid bounded by a plane wall (the case $\Lambda = 0 = \Omega^*$) shows that the relative velocity components appear in the leading terms, as expected, only [5.6b] has the Faxen law term (order ϵ^2) and both have an $O(\epsilon^3)$ contribution arising from the rate of change of the imposed axisymmetrical flow **W**.

Further, a closer examination of the $O(\epsilon^3)$ terms in formula [5.5] shows that they depend on the stagnation flow and the sphere's translation as well as the geometry of the flow boundaries. The $O(\epsilon^3)$ terms on the right hand side of [3.21] can be written down from [3.17] after noting that $D_3(\epsilon) = O(\epsilon^4)$ and taking account of the odd and even functions of $(x - x_0)$. Thus [3.21] has the form

$$\frac{L}{8\pi\mu\epsilon^3}=-\frac{E(\epsilon)}{\epsilon^3}=-\frac{1}{2}\Lambda x_0-\Omega^*-\frac{1}{2}C_1(\epsilon)\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)-\frac{1}{2}E(\epsilon)\left(\frac{\partial\Omega_x}{\partial z}-\frac{\partial\Omega_z}{\partial x}\right).+O(\epsilon^4)$$

where $C_1(\epsilon)=\frac{1}{4}\epsilon^3(-\Lambda x_0z_0-U^*)$. Then, by observing that $\nabla^2\mathbf{U}$ is conservative, the expressions (5.4) suffice to show that a more accurate form of [5.5] is

$$\frac{L}{8\pi\mu\epsilon^3}\sim\frac{-\dfrac{1}{2}\Lambda x_0-\Omega^*+\dfrac{3}{32}(\epsilon/z_0)^3(\Lambda x_0z_0+U^*)/z_0}{1-\dfrac{5}{16}(\epsilon/z_0)^3} \qquad [5.7]$$

where the denominator correction is geometrical whilst that in the numerator indicates the creation of vorticity at $(x_0, 0, z_0)$ by the dipole solution excited by the relative velocity component $(-\Lambda x_0z_0 - U^*)$ parallel to the plane. The axisymmetric geometry precludes a similar contribution from the normal component $(\Lambda z_0^2 - V^*)$.

Table 1.

ξ	1.0	1.5	2.0	2.5	3.0
$\epsilon/z_0 = \operatorname{sech}\xi$	0.6481	0.4251	0.2658	0.1631	0.0993
$F_x/6\pi\mu U^*\epsilon$	−1.4937	−1.2979	−1.1726	−1.1003	−1.0590
O'Neill (1964)	−1.5675	−1.3079	−1.1738	−1.1006	−1.0591
$F_z/6\pi\mu V^*\epsilon$	−2.4569	−1.7852	−1.4077	−1.2214	−1.1252
Brenner (1961)	−3.0361	−1.8375	−1.4129	−1.2220	−1.1252

The accuracy of the third order truncation can be assessed in this case by comparison of [5.6a, b] with the corresponding exact solutions in bispherical coordinates given by O'Neill (1964) and Brenner (1961) respectively for a moving sphere in a quiescent fluid. On setting $\Lambda = 0$, the numerical values obtained are displayed in table 1 and show, as might be expected, closer agreement for F_x than for F_z.

6. POISEUILLE FLOW

Here S is the cylinder $\rho = 1$ so the velocity fields $\mathbf{\Omega}$, $\mathbf{U}$, $\mathbf{u}$, $\mathbf{V}$ and $\mathbf{v}$ evidently depend on x, x_0, y and $(z - z_0)$ only, i.e. ρ, ρ_0, ω and $(z - z_0)$ in the notation of [4.5]. The boundary conditions [3.14] imply that Ω_z, U_x, u_x, V_z and v_z are even functions of $(z - z_0)$ whilst Ω_x, U_z, u_z, V_x and v_x are odd functions of $(z - z_0)$. When these properties and the Poiseuille flow $\mathbf{W} = \frac{1}{4}G(1 - \rho^2)\hat{z}$ are substituted into [3.20] and then [3.19a, b], it follows that

$$Q_x = 0, \qquad Q_z = 0$$

and hence, on using [4.6] and the symmetry of U_x and V_z, the equations for F_x and F_z again separate to yield:

$$\frac{F_x}{6\pi\mu\epsilon}\left[1 - \frac{3}{4}\epsilon U_x - \frac{1}{4}\epsilon^3\nabla^2 U_x\right] = -U^* \qquad [6.1a]$$

$$\frac{F_z}{6\pi\mu\epsilon}\left[1 - \frac{3}{4}\epsilon V_z - \frac{1}{4}\epsilon^3\nabla^2 V_z\right] = \frac{1}{4}G(1 - \rho_0^2) - V^* - \frac{1}{6}\epsilon^2 G + \epsilon^3\left(\frac{1}{4}G\rho_0 - \Omega^*\right)\Omega_z. \qquad [6.1b]$$

Also [3.21] takes the form

$$\frac{L}{8\pi\mu\epsilon^3} = \frac{1}{4}G\rho_0 - \Omega^* + \frac{3}{8}\epsilon\left(1 + \frac{3}{4}\epsilon V_z\right)\left[\frac{1}{4}G(1 - \rho_0^2) - V^*\right]\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + O(\epsilon^3). \qquad [6.2]$$

Equation [6.1a] shows that F_x vanishes when U^* is zero, as expected because the reversibility of this Stokes flow indicates that the sphere can be neither attracted to nor repelled from the cylinder wall.

The solution for the Stokes flow in a pipe due to an arbitrarily placed and orientated stokeslet has been given by Liron & Shahar (1978) in terms of the velocity representation used by Happel & Brenner (1973, p. 77), so only the following brief description of an alternate more direct but novel solution, which displays the symmetry of V_z and U_ρ, is included here. Such symmetry is predicted by applying Green's theorem, as in section 4, to the fields generated by two stokeslets placed in the $\hat{\rho}$ direction(s) at (ρ, ω, z) and $(\rho_0, 0, z_0)$.

The boundary conditions on $\mathbf{U}$ and $\mathbf{V}$ are, from [3.14],

$$\left.\begin{aligned}\mathbf{U} &= \frac{1}{R_1}\hat{x} + \frac{\cos\omega - \rho_0}{R_1^3}[\hat{\rho} - \rho_0\hat{x} + (z - z_0)\hat{z}]\\ \mathbf{V} &= \frac{1}{R_1}\hat{z} + \frac{z - z_0}{R_1^3}[\hat{\rho} - \rho_0\hat{x} + (z - z_0)\hat{z}]\end{aligned}\right\}\ \text{at } \rho = 1. \qquad [6.3]$$

where

$$\frac{1}{R_1} = [1 + \rho_0^2 - 2\rho_0\cos\omega + (z - z_0)^2]^{-(1/2)}$$
$$= \frac{2}{\pi}\int_0^\infty K_0[\lambda\sqrt{(1 + \rho_0^2 - 2\rho_0\cos\omega)}]\cos\lambda(z - z_0)\,d\lambda.$$

These suggest the introduction of the Fourier transforms

$$\mathbf{U} = \frac{2}{\pi}\int_0^\infty [(\bar{U}_\rho\hat{\rho} + \bar{U}_\omega\hat{\omega})\cos\lambda(z - z_0) - \bar{U}_z\hat{z}\sin\lambda(z - z_0)]\,d\lambda$$

$$\mathbf{V} = \frac{2}{\pi}\int_0^\infty [(\bar{V}_\rho\hat{\rho} + \bar{V}_\omega\hat{\omega})\sin\lambda(z - z_0) + \bar{V}_z\hat{z}\cos\lambda(z - z_0)]\,d\lambda \qquad [6.4]$$

with corresponding pressures

$$p_u = \frac{2}{\pi}\int_0^\infty \bar{p}_u\cos\lambda(z - z_0)\,d\lambda, \qquad p_v = \frac{2}{\pi}\int_0^\infty \bar{p}_v\sin\lambda(z - z_0)\,d\lambda.$$

Then the calculation can be reduced to that of four transformed fields

$$\{(u^{(j)}\hat{\rho} + v^{(j)}\hat{\omega} + w^{(j)}\hat{z});\qquad j = 1, 2, 3, 4\}$$

such that

$$(u^{(j)}\hat{\rho} + v^{(j)}\hat{\omega} + w^{(j)}\hat{z})_{\rho=1} = \left[\hat{x}\delta_{1j} + \hat{\rho}\delta_{2j} + \hat{z}\left(\delta_{3j} + \delta_{4j}\frac{\partial}{\partial\lambda}\right)\right]K_0[\lambda\sqrt{(1 + \rho_0^2 - 2\rho_0\cos\omega)}], \qquad [6.5]$$

where δ_{ij} is the Kronecker delta, since then

$$\bar{V}_z = \lambda(w^{(2)} - \rho_0 w^{(1)} + w^{(4)}) + 2w^{(3)}$$

$$\bar{U}_\rho = \frac{\partial}{\partial\rho_0}(u^{(2)} - \rho_0 u^{(1)} + u^{(4)}) + 2u^{(1)} \qquad [6.6]$$

with corresponding expressions for $\bar{V}_\rho$, $\bar{V}_\omega$, $\bar{p}_v$, $\bar{U}_\omega$, $\bar{U}_z$ and $\bar{p}_u$. On writing

$$\begin{Bmatrix} u^{(j)} \\ w^{(j)} \\ p^{(j)} \end{Bmatrix} = \sum_{m=0}^{\infty} \epsilon_m \begin{Bmatrix} u_m^{(j)} \\ w_m^{(j)} \\ p_m^{(j)} \end{Bmatrix}\cos m\omega, \qquad v^{(j)} = 2\sum_{m=1}^{\infty} v_m^{(j)}\sin m\omega \qquad [6.7]$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m \geqslant 1)$, it follows that since

$$K_0[\lambda\sqrt{(1 + \rho_0^2 - 2\rho_0\cos\omega)}] = \sum_{m=0}^{\infty}\epsilon_m K_m(\lambda)I_m(\lambda\rho_0)\cos m\omega \qquad (\rho_0 < 1),$$

the boundary conditions [6.3] imply that the Fourier components are such that

$$[u_{m-1}^{(1)} + v_{m-1}^{(1)}]_{\rho=1} = K_m(\lambda)I_m(\lambda\rho_0); \qquad [v_m^{(j)}]_{\rho=1} = 0 \qquad (j = 2, 3, 4) \quad (m \geqslant 1)$$

$$\left.\begin{aligned} &[u_{m+1}^{(1)} - v_{m+1}^{(1)}]_{\rho=1} = K_m(\lambda)I_m(\lambda\rho_0) = [u_m^{(2)}]_{\rho=1} = [w_m^{(3)}]_{\rho=1} \\ &[w_m^{(1)}]_{\rho=1} = [w_m^{(2)}]_{\rho=1} = [u_m^{(3)}]_{\rho=1} = [u_m^{(4)}]_{\rho=1} = 0 \\ &[w_m^{(4)}]_{\rho=1} = \frac{\partial}{\partial\lambda}[K_m(\lambda)I_m(\lambda\rho_0)] \end{aligned}\right\} \quad (m \geqslant 0). \qquad [6.8]$$

These fields are now determined by applying the creeping flow equations [2.1] which yield, for each j (suffix omitted for convenience) and each $m \geqslant 0$:

$$\frac{\partial u_m}{\partial \rho} + \frac{u_m + m v_m}{\rho} - \lambda w_m = 0$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \lambda^2 - \frac{m^2}{\rho^2}\right] w_m = \mu^{-1}\lambda p_m$$

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \lambda^2 - \frac{(m \pm 1)^2}{\rho^2}\right](u_m \pm v_m) = \mu^{-1}\left[\frac{dp_m}{d\rho} \mp \frac{m}{\rho} p_m\right]$$

Now since the fluid pressure satisfies Laplace's equation, it follows that

$$p_m = 2\mu q_m(\lambda) I_m(\lambda\rho) \qquad (m \geqslant 0)$$

whence

$$\frac{dp_m}{d\rho} \mp \frac{m}{\rho} p_m = 2\mu\lambda q_m(\lambda) I_{m\pm 1}(\lambda\rho).$$

Thus the equations for w_m and $(u_m \pm v_m)$ are of the same form and have solutions

$$w_m = q_m(\lambda)\rho I_{m+1}(\lambda\rho) + \frac{1}{\lambda} r_m(\lambda) I_m(\lambda\rho) \qquad (m \geqslant 0)$$

$$u_m \pm v_m = q_m(\lambda)\rho I_{m+1\pm 1}(\lambda\rho) + \frac{1}{\lambda} s_m{}^{\pm}(\lambda) I_{m\pm 1}(\lambda\rho) \qquad (m \geqslant 1)$$

$$u_0 = q_0(\lambda)\rho I_2(\lambda\rho) + \frac{1}{\lambda} s_0(\lambda) I_1(\lambda\rho).$$

The continuity equation shows that

$$s_0(\lambda) = r_0(\lambda); \qquad 2q_m(\lambda) + s_m{}^{+}(\lambda) + s_m{}^{-}(\lambda) - 2r_m(\lambda) = 0 \qquad (m \geqslant 1).$$

Hence, if $m = 0$, the unknown functions $q_0(\lambda)$ and $r_0(\lambda)$ are given by

$$q_0(\lambda) I_2(\lambda) + \frac{1}{\lambda} r_0(\lambda) I_1(\lambda) = (u_0)_{\rho=1}$$

$$q_0(\lambda) I_1(\lambda) + \frac{1}{\lambda} r_0(\lambda) I_0(\lambda) = (w_0)_{\rho=1}$$

whilst, if $m \geqslant 1$, the four functions $q_m(\lambda)$, $s_m{}^{\pm}(\lambda)$ and $r_m(\lambda)$ are given by

$$\begin{aligned}
&q_m(\lambda) I_{m+2}(\lambda) + \frac{1}{\lambda} s_m{}^{+}(\lambda) I_{m+1}(\lambda) &&= (u_m + v_m)_{\rho=1}\\
&q_m(\lambda) I_m(\lambda) + \frac{1}{\lambda} s_m{}^{-}(\lambda) I_{m-1}(\lambda) &&= (u_m - v_m)_{\rho=1}\\
&q_m(\lambda) I_{m+1}(\lambda) + \frac{1}{\lambda} r_m(\lambda) I_m(\lambda) &&= (w_m)_{\rho=1}\\
&2q_m(\lambda) + s_m{}^{+}(\lambda) + s_m{}^{-}(\lambda) - 2r_m(\lambda) &&= 0.
\end{aligned}$$

Here the determinant of coefficients is $\lambda^{-3}\Delta_m(\lambda)$, where Δ_m has the same definition as in [4.14] of Liron & Shahar (1978) after setting their R_0 equal to unity. The alternative expression

$$\Delta_m(\lambda) = 2\lambda I_{m-1}(I_m I_{m+2} - I_{m+1}^2) + 2mI_m(I_{m-1}I_{m+1} - I_m^2) \qquad [6.9]$$

where each modified Bessel function is evaluated at λ, shows clearly the relationship to the $m = 0$ case and is of assistance in computation.

On solving the above sets of equations for each m and j, with the r.h.s. given by [6.8], it follows on substitution in [6.7] and then [6.6] that

$$\bar{V}_z = \frac{1}{I_1^2 - I_0 I_2}\{\rho\rho_0 I_1(\lambda\rho)I_1(\lambda\rho_0) - \lambda(I_2K_0 + I_1K_1)[\rho I_1(\lambda\rho)I_0(\lambda\rho_0) + \rho_0 I_1(\lambda\rho_0)I_0(\lambda\rho)]$$
$$+ [1 - 2(I_2K_0 + I_1K_1)]I_0(\lambda\rho)I_0(\lambda\rho_0)\}$$
$$- \sum_{m=1}^{\infty}\frac{2\lambda}{\Delta_m(\lambda)}\cos m\omega\left\{(I_{m-1} + I_{m+1})\rho\rho_0 I_{m+1}(\lambda\rho)I_{m+1}(\lambda\rho_0)\right.$$
$$- 2\lambda\left[I_{m-1}(I_{m+2}K_m + I_{m+1}K_{m+1}) - \frac{m}{\lambda}I_m(I_{m+1}K_{m-1} + I_mK_m)\right][\rho I_{m+1}(\lambda\rho)I_m(\lambda\rho_0)$$
$$+ \rho_0 I_{m+1}(\lambda\rho_0)I_m(\lambda\rho)] + \Big[\!\!\Big[(I_{m-1} + I_{m+1})[1 - 2m(I_{m+1}K_{m-1} + I_mK_m)]$$
$$\left. - 4\left[I_{m-1}(I_{m+2}K_m + I_{m+1}K_{m+1}) - \frac{m}{\lambda}K_m(I_{m+1}I_{m-1} + I_m^2)\right]\Big]\!\!\Big] I_m(\lambda\rho)I_m(\lambda\rho_0)\right\}$$

after suitable manipulation by means of the recurrence relations and the identities

$$I_mK_{m-1} + I_{m-1}K_m = \frac{1}{\lambda}, \qquad I_{m-1}K_{m+1} - I_{m+1}K_{m-1} = 2m/\lambda^2 \qquad (m \geq 1).$$

Also, with $\Delta_m(\lambda)$ defined by [6.9],

$$\bar{U}_\rho = (I_1^2 - I_0I_2)^{-1}[\rho I_2(\lambda\rho), I_1(\lambda\rho)]\underset{\sim}{A}^{(0)}\begin{bmatrix}\rho_0 I_2(\lambda\rho_0)\\ I_1(\lambda\rho_0)\end{bmatrix}$$
$$- 2\sum_{m=1}^{\infty}\frac{\lambda}{\Delta_m(\lambda)}[\rho I_{m+2}(\lambda\rho), I_{m+1}(\lambda\rho), I_{m-1}(\lambda\rho)]\ \underset{\sim}{A}^{(m)}\begin{bmatrix}\rho_0 I_{m+2}(\lambda\rho_0)\\ I_{m+1}(\lambda\rho_0)\\ I_{m-1}(\lambda\rho_0)\end{bmatrix}\cos m\omega$$

where $\underset{\sim}{A}^{(0)}$ is a 2×2 matrix of coefficients which is identical to that appearing in the corresponding Fourier component of $\bar{V}_z$ and, for each $m \geq 1$, $\underset{\sim}{A}^{(m)}$ is a symmetric 3×3 matrix of coefficients, whose details are omitted here. Thus, with U_ρ and V_z determined from $\bar{U}_\rho$, $\bar{V}_z$ by [6.4], it is seen that the symmetric form of U_ρ involves more functions of

ρ than that for V_z. The first order coefficients U_x, V_z in [6.1a, b] are given by

$$U_x = \frac{2}{\pi}\int_0^\infty (\bar{U}_\rho)_{\substack{\rho=\rho_0\\ \omega=0}} \mathrm{d}\lambda, \qquad V_z = \frac{2}{\pi}\int_0^\infty (\bar{V}_z)_{\substack{\rho=\rho_0\\ \omega=0}} \mathrm{d}\lambda \tag{6.10}$$

and other quantities in these equations for F_x, F_z are in principle determined by similar calculations. Liron & Shahar (1978) showed that, for $z \neq z_0$, the integrals for U and V in [6.4] can be expressed as series in which the terms decay exponentially with $|z - z_0|$ and exploited this computational advantage to display several profiles of $\mathbf{V}\cdot\hat{z}$, principally for the axisymmetric case $\rho_0 = 0$. This information is not helpful to the current considerations which require values of the expressions [6.10] and others for $0 \leqslant \rho_0 < 1$. These integrals converge rapidly without the need for prior rearrangements and values of V_z for various ρ_0 are given by Happel & Brenner (1973).

In the axisymmetric case $\rho_0 = 0$, the formula for $\bar{V}_z$ simplifies to that given by Sonshine, Cox & Brenner (1966) and, by symmetry, Ω_z vanishes in [6.1b]. After the numerical evaluation of two integrals, this force formula becomes

$$\frac{F_z}{6\pi\mu\epsilon} \sim \frac{\frac{1}{4}G - V^* - \frac{1}{6}\epsilon^2 G}{1 - 2.10443\epsilon + 2.0888\epsilon^3}$$

where the first order correction is the wall correction factor quoted by Brenner (1962) and the third order coefficient is quoted from Happel & Brenner (1973, p. 318).

7. FLOW PAST A SPHERE

Here S is the sphere $R = 1$, where $\rho = R\sin\alpha$, $z = R\cos\alpha$, and the axisymmetric flow W is given by

$$\mathbf{W} = -\frac{3}{4}Uz\left(\frac{1}{R^2} - \frac{1}{R^4}\right)\hat{R} + U\left(1 - \frac{3}{4R} - \frac{1}{4R^3}\right)\hat{z} \tag{7.1}$$

which, when substituted into (3.20), yields

$$\mathbf{Q} = -\frac{1}{2}Uz_0\left(\frac{3}{R_0^5} - \frac{5}{R_0^7}\right)\left[x_0^2\frac{\partial \mathbf{U}}{\partial x_0} + x_0 z_0\left(\frac{\partial \mathbf{U}}{\partial z_0} + \frac{\partial \mathbf{V}}{\partial x_0}\right) + z_0^2\frac{\partial \mathbf{V}}{\partial z_0}\right]$$
$$- \frac{U}{R_0^5}\left[\frac{1}{2}x_0\left(\frac{\partial \mathbf{U}}{\partial z_0} + \frac{\partial \mathbf{V}}{\partial x_0}\right) + z_0\frac{\partial \mathbf{V}}{\partial z_0}\right] \tag{7.2}$$

is an obvious notation.

The spherically symmetric geometry can be exploited by writing the fields U, V as sums of resolved components of the reflected velocities $-\hat{\mathbf{V}}$, $-\hat{\mathbf{U}}$ generated by the sphere when stokeslets in the radial ($\hat{R}$) and transverse ($\hat{\alpha}$) directions respectively are placed by $(x_0, 0, z_0)$, where $R_0 = (x_0^2 + z_0^2)^{1/2} > 1$. Then

$$\mathbf{U} = \frac{z_0}{R_0}\hat{\mathbf{U}} + \frac{x_0}{R_0}\hat{\mathbf{V}}, \qquad \mathbf{V} = -\frac{x_0}{R_0}\hat{\mathbf{U}} + \frac{z_0}{R_0}\hat{\mathbf{V}} \tag{7.3}$$

at all points of the fluid and the behaviour of U, V near $(x_0, 0, z_0)$ can be deduced from the simpler case $x_0 = 0$. The fields **u** and **v** can be similarly written in terms of reflected velocities $-\mathring{\mathbf{u}}$, $-\mathring{\mathbf{v}}$ arising from radially and transversely directed dipoles. The tensorial character of [3.19a, b] and [3.21] enables these to be written in the form

$$\frac{\mathbf{F}}{6\pi\mu\epsilon}+\frac{F_R}{6\pi\mu\epsilon}\left(-\frac{3}{4}\epsilon\mathring{\mathbf{V}}+\frac{1}{4}\epsilon^3\mathring{\mathbf{v}}-\frac{1}{8}\epsilon^3\nabla^2\mathring{\mathbf{V}}\right)+\frac{F_x}{6\pi\mu\epsilon}\left(-\frac{3}{4}\epsilon\mathring{\mathbf{U}}+\frac{1}{4}\epsilon^3\mathring{\mathbf{u}}-\frac{1}{8}\epsilon^3\nabla^2\mathring{\mathbf{U}}\right)$$

$$=\mathbf{W}-(U^*\hat{x}+V^*\hat{z})+\frac{1}{6}\epsilon^2\nabla^2\mathbf{W}+\epsilon^3\left(\frac{1}{2}\hat{\omega}\cdot\operatorname{curl}\mathbf{W}-\Omega^*\right)\mathbf{\Omega}-\frac{5}{4}\epsilon^3\mathbf{Q} \quad [7.4]$$

$$\frac{L}{8\pi\mu\epsilon^3}=\frac{1}{2}\hat{\omega}\cdot\operatorname{curl}\mathbf{W}-\Omega^*+\frac{F_R}{16\pi\mu}\hat{\omega}\cdot\operatorname{curl}\mathring{\mathbf{V}}+\frac{F_x}{16\pi\mu}\hat{\omega}\cdot\operatorname{curl}\mathring{\mathbf{U}}+O(\epsilon^3) \quad [7.5]$$

whilst the substitution of (7.3) into (7.2) eventually yields

$$\mathbf{Q}=\frac{U}{2R_0^5}\left[-3z_0(R_0^2-1)\frac{\partial\mathring{\mathbf{V}}}{\partial R_0}+x_0\left(\frac{\partial\mathring{\mathbf{U}}}{\partial R_0}+\frac{1}{R_0}\frac{\partial\mathring{\mathbf{V}}}{\partial\alpha_0}-\frac{\mathring{\mathbf{U}}}{R_0}\right)\right] \quad [7.6a]$$

Similarly (3.15) can be written

$$\mathbf{\Omega}=\frac{1}{2}\left(\frac{\partial\mathring{\mathbf{U}}}{\partial R_0}-\frac{1}{R_0}\frac{\partial\mathring{\mathbf{V}}}{\partial\alpha_0}+\frac{\mathring{\mathbf{U}}}{R_0}\right). \quad [7.6b]$$

The required components of the reflected velocities $\mathring{\mathbf{U}}$ and $\mathring{\mathbf{V}}$ and their derivatives at $(x_0, 0, z_0)=(R_0\sin\alpha_0, 0, R_0\cos\alpha_0)$ can be calculated from knowledge of $\mathbf{U}$ and $\mathbf{V}$ in the neighbourhood of the stokeslets when these are placed on the z-axis at $(0, 0, R_0)$. Evidently the l.h.s. of (7.4) can be written in the separated form

$$\frac{F_R}{6\pi\mu\epsilon}\left(1-\frac{3}{4}\epsilon V_z-\frac{1}{4}\epsilon^3\nabla^2V_z\right)_{\substack{x_0=0\\z_0=R_0}}+\frac{F_x}{6\pi\mu\epsilon}\left(1-\frac{3}{4}\epsilon U_x-\frac{1}{4}\epsilon^3\nabla^2U_x\right)_{\substack{x_0=0\\z_0=R_0}}$$

after invoking [4.6] and the symmetry of U_x and V_z with respect to field and stokeslet positions. Also [7.5] and [7.6a, b] can be written

$$\frac{L}{8\pi\mu\epsilon^3}=\frac{1}{2}\hat{\omega}\cdot\operatorname{curl}\mathbf{W}-\Omega^*+\frac{F_x}{16\pi\mu}(\hat{y}\cdot\operatorname{curl}\mathbf{U})_{\substack{x_0=0\\z_0=R_0}}+O(\epsilon^3)$$

$$\mathbf{Q}=-\frac{3Uz_0}{2R_0^5}(R_0^2-1)\left(\frac{\partial V_z}{\partial z_0}\right)_{\substack{x_0=0\\z_0=R_0}}\hat{R}+\frac{Ux_0}{2R_0^5}\left(\frac{\partial U_x}{\partial z_0}+\frac{\partial V_x}{\partial x_0}-\frac{U_x}{z_0}\right)_{\substack{x_0=0\\z_0=R_0}}\hat{\alpha}$$

$$\mathbf{\Omega}=\frac{1}{2}\left(\frac{\partial U_x}{\partial z_0}-\frac{\partial V_x}{\partial x_0}+\frac{U_x}{z_0}\right)_{\substack{x_0=0\\z_0=R_0}}\hat{\alpha}$$

where it is still understood that (x, y, z) is set equal to $(x_0, 0, z_0)$ after all differentiations have been completed.

In the axisymmetric case, an elementary calculation shows that

$$\mathbf{V}(\rho, z; 0, z_0)=\operatorname{curl}\left\{\frac{\frac{1}{2}\rho\hat{\omega}}{[\rho^2z_0^2+(zz_0-1)^2]^{1/2}}\left[3-\frac{\rho^2+(z-z_0)^2}{\rho^2z_0^2+(zz_0-1)^2}\right]\right\},$$

i.e. in the above expressions,

$$\left(V_z, \nabla^2 V, \frac{\partial V_z}{\partial z_0}, \frac{\partial V_x}{\partial x_0}\right)_{\substack{x_0=0\\ z_0=R_0}} = \left(\frac{3}{R_0^2-1}, \frac{-2(3R_0^2+5)}{(R_0^2-1)^3}, \frac{-3R_0}{(R_0^2-1)^2}, \frac{-3R_0}{2(R_0^2-1)^2}\right).$$

For the asymmetric case, an appropriate velocity representation is that used by Ranger (1973) for flow past a spherical cap, namely

$$\mathbf{U}(\rho, \omega, z; 0, z_0) = \text{curl}^2\left[\frac{\psi}{\sin\alpha}\hat{R}\cos\omega\right] + \text{curl}\left[\frac{\chi}{\sin\alpha}\hat{R}\sin\omega\right]$$

where

$$\psi = \psi_1 - \frac{1}{2}(R^2-1)\left(R\frac{\partial\psi_1}{\partial R} - \psi_2\right)$$

and solutions for ψ_1, ψ_2 and χ which vanish as $R \to \infty$ are

$$(\psi_1, \psi_2, \chi) = \sum_{n=1}^{\infty}(A_n, B_n, C_n)R^{-n}P_n'(\cos\alpha)\sin^2\alpha.$$

The coefficients are determined by application of the boundary conditions at $R = 1$, whence it follows eventually that the required quantities are given by

$$\left(U_x, \nabla^2 U_x, \hat{y}\cdot\text{curl}\,\mathbf{U}, \frac{\partial U_x}{\partial z_0}\right)_{\substack{x_0=0\\ z_0=R_0}} = \left[\frac{3}{2}\left(\frac{1}{R_0^2-1} - \frac{1}{2R_0^2}\right),\right.$$

$$\left.\frac{1}{R_0^2-1}\left(\frac{-4}{(R_0^2-1)^2} + \frac{3}{R_0^2-1} - \frac{3}{2R_0^2}\right), \frac{-3}{2R_0(R_0^2-1)}, \frac{3}{2}\left(\frac{-R_0}{(R_0^2-1)^2} + \frac{1}{2R_0^3}\right)\right].$$

When these results and [7.1] are substituted into [7.4] and [7.5], the $\hat{R}$ and $\hat{\alpha}$ components of the effective Stokes relative velocity are given by the separate formulae

$$\frac{F_R}{6\pi\mu\epsilon} \sim \left\{\frac{z_0}{R_0}\left[U\left(1 - \frac{3}{2R_0} + \frac{1}{2R_0^3}\right) - V^*\right] - \frac{x_0}{R_0}U^* + \frac{\epsilon^2 U z_0}{2R_0^4} - \frac{45\epsilon^3 U z_0}{8R_0^4(R_0^2-1)}\right\}$$
$$\div\left[1 - \frac{9\epsilon}{4(R_0^2-1)} + \frac{\epsilon^3(3R_0^2+5)}{2(R_0^2-1)^3}\right] \qquad [7.7a]$$

$$\frac{F_\alpha}{6\pi\mu\epsilon} \sim \left\{-\frac{x_0}{R_0}\left[U\left(1 - \frac{3}{4R_0} - \frac{1}{4R_0^3}\right) - V^*\right] - \frac{z_0}{R_0}U^* + \frac{\epsilon^2 U x_0}{4R_0^4}\right.$$
$$\left. - \frac{3\epsilon^3}{4R_0(R_0^2-1)}\left(\frac{3Ux_0}{4R_0^3} + \Omega^*\right) + \frac{15\epsilon^3 U x_0}{8R_0^6(R_0^2-1)}\left[\frac{R_0^2}{R_0^2-1} + \frac{1}{2R_0^2}\right]\right\}$$
$$\div\left[1 - \frac{9}{8}\epsilon\left(\frac{1}{R_0^2-1} - \frac{1}{2R_0^2}\right) + \frac{\epsilon^3}{R_0^2-1}\left(\frac{1}{(R_0^2-1)^2} - \frac{3}{4(R_0^2-1)} + \frac{3}{8R_0^2}\right)\right]$$

[7.7b]

and the torque L by

$$\frac{L}{8\pi\mu\epsilon^3} = -\left(\frac{3Ux_0}{4R_0^3} + \Omega^*\right) + \frac{9\epsilon\left\{x_0\left[U\left(1 - \frac{3}{4R_0} - \frac{1}{4R_0^3}\right) - V^*\right] + z_0U^*\right\}}{16R_0^2(R_0^2 - 1)\left[1 - \frac{9}{8}\epsilon\left(\frac{1}{R_0^2 - 1} - \frac{1}{2R_0^2}\right)\right]} + O(\epsilon^3) \qquad [7.8]$$

Again the accuracy of the third order truncation can be assessed because an exact solution for the axisymmetric flow is available from the analyses of Stimson & Jeffrey (1926) and Davis (1978). These provide a comparison for the force coefficient estimates obtained by setting $U^* = 0 = x_0$ in [7.7a], viz

$$\frac{F_z}{6\pi\mu\epsilon} \sim \frac{U\left(1 - \frac{3}{2z_0} + \frac{1}{2z_0^3}\right) - V^* + \frac{\epsilon^2 U}{2z_0^3} - \frac{45\epsilon^3 U}{8z_0^3(z_0^2 - 1)}}{1 - \frac{9\epsilon}{4(z_0^2 - 1)} + \frac{\epsilon^3(3z_0^2 + 5)}{2(z_0^2 - 1)^3}}. \qquad [7.9]$$

Values are displayed in table 2 for the independent cases of a moving sphere in quiescent fluid ($U = 0$) and a fixed sphere in streaming flow ($V^* = 0$), both in the presence of a fixed larger sphere. Agreement is good whenever the gap between spheres exceeds the smaller radius ($z_0 - 1 > 2\epsilon$). The smaller values in the last two columns are due to the smaller sphere lying in the vicinity of a stagnation point on the larger sphere.

Table 2.

Values of ξ at spheres		ϵ	z_0	$F_z/6\pi\mu V^*\epsilon$ when $U = 0$ [7.9]	Exact	$F_z/6\pi\mu U\epsilon$ when $V^* = 0$ [7.9]	Exact
−0.1,	3	0.0099	1.106	−1.11120	−1.11205	0.014772	0.014771
−0.1,	2	0.0276	1.109	−1.3569	−1.3608	0.01890	0.01894
−0.1,	1.5	0.0470	1.116	−1.6709	−1.7094	0.02472	0.02533
−0.1,	1	0.0852	1.137	−2.2063	−2.6002	0.03495	0.04287
−0.3,	3	0.0304	1.351	−1.0899	−1.0900	0.10108	0.10107
−0.3,	2	0.0840	1.361	−1.2763	−1.2786	0.1229	0.1230
−0.3,	1.5	0.1430	1.382	−1.4993	−1.5210	0.1523	0.1543
−0.3,	1	0.2591	1.445	−1.8505	−2.0481	0.2067	0.2293
−0.5,	3	0.0520	1.651	−1.07250	−1.07252	0.21759	0.21758
−0.5,	2	0.1437	1.668	−1.2163	−1.2178	0.2541	0.2542
−0.5,	1.5	0.2447	1.703	−1.3793	−1.3920	0.3015	0.3037
−0.5,	1	0.4434	1.812	−1.6198	−1.7266	0.3884	0.4113
−1,	2.5	0.1942	2.734	−1.0719	−1.0720	0.510780	0.510780
−1,	2	0.3240	2.762	−1.1215	−1.1220	0.5405	0.5406
−1,	1.5	0.5519	2.841	−1.2028	−1.2065	0.5957	0.5969

Evidently in the region $\rho^2 + (z - 1)^2 = O(\delta^2)$ where $\delta \ll 1$, the flow field $\mathbf{W}$ has velocities of order $U\delta^2$ and to this order is a stagnation flow at a plane. Indeed, on introducing the rescaling

$$x_0 = \delta\hat{x}_0, \qquad z_0 = 1 + \delta\hat{z}_0, \qquad \epsilon = \delta\hat{\epsilon}, \qquad \frac{3}{2}U\delta^2 = \Lambda, \qquad \Omega^*\delta = \hat{\Omega}^*$$

$$F_R = \delta\hat{F}_z, \qquad F_x = \delta\hat{F}_x, \qquad L = \delta^2\hat{L}$$

into the formulae [7.7a, b], [7.8] and letting $\delta \to 0$, the corresponding stagnation flow formulae [5.6a, b] and [5.5] are recovered.

8. FLOW THROUGH A HOLE IN A PLANE

Here S is the region $\rho > 1$ of the plane $z = 0$ and $\mathbf{W}$ is the pressure driven flow described by Happel & Brenner (1973, section 4.29). Thus

$$\mathbf{W} = U\frac{(\sin\nu \sinh\lambda\,\hat{\rho} + \cos\nu\cosh\lambda\,\hat{z})\cos^2\nu}{(\sinh^2\lambda + \cos^2\nu)\cosh\lambda} \qquad [8.1a]$$

where λ, ν are elliptic coordinates defined by $\rho = \cosh\lambda\sin\nu$, $z = \sinh\lambda\cos\nu$ ($0 \leqslant \nu \leqslant \pi/2$, $-\infty < \lambda < \infty$) and $U\hat{z}$ is the velocity at the origin, i.e. the center of the orifice. The corresponding pressure p_w is given by

$$p_w = -2\mu U\left[\frac{\sinh\lambda}{\sinh^2\lambda + \cos^2\nu} + \tan^{-1}(\sinh\lambda)\right] \qquad [8.1b]$$

In particular

$$\mathbf{W}(0, z) = \frac{U\hat{z}}{z^2+1}, \qquad p_w(0, z) = -2\mu U\left(\frac{z}{z^2+1} + \tan^{-1}z\right)$$

and hence the axisymmetric formula [3.23] becomes

$$\frac{F_z}{6\pi\mu\epsilon} \sim \left\{-V^* + \frac{U}{(z_0^2+1)^2}\left[z_0^2 + 1 - \frac{2}{3}\epsilon^2 - \frac{5}{2}\epsilon^3 z_0\frac{\partial V_z}{\partial z_0}\right]\right\}$$

$$\div\left[1 - \frac{3}{4}\epsilon V_z + \frac{1}{4}\epsilon^3\left(v_z - \frac{1}{2}\nabla^2 V_z\right)\right] \qquad [8.2]$$

Now, according to Davis, O'Neill & Brenner (1981), the reflected velocity in this axisymmetric case is such that

$$\mathbf{V}\cdot\hat{z} = \left(z\frac{\partial}{\partial z} - 1\right)\left(z_0\frac{\partial}{\partial z_0} - 1\right)F + zz_0G \qquad [8.3]$$

where F and G are harmonic functions of ρ, z given by

$$F(\rho, z; z_0) = \frac{2}{\pi}\int_0^\infty e^{-k|z|}J_0(k\rho)\int_1^\infty \frac{s\sin ks}{z_0^2+s^2}\,ds\,dk \qquad [8.4a]$$

$$G(\rho, z; z_0) = \frac{2}{\pi}\int_0^\infty e^{-k|z|}J_0(k\rho)\left[\frac{\sin k}{z_0^2+1} + k\int_1^\infty \frac{z_0^2 - s^2}{(z_0^2+s^2)^2}\cos ks\,ds\right]dk. \qquad [8.4b]$$

The reflected velocity due to the corresponding dipole singularity can be similarly shown to have $\hat{z}$-component

$$\mathbf{v}\cdot\hat{z} = -\left(z\frac{\partial}{\partial z} - 1\right)\frac{\partial^2 F}{\partial z_0^2} - z\frac{\partial G}{\partial z_0}$$

and hence, since $F(0, z; z_0)$ and $G(0, z; z_0)$ are evidently symmetric, it follows from [8.3] that $v_z = -\frac{1}{2}\nabla^2 V_z$ in the force formula [8.2] as predicted in section 4. Further, by observing that [8.3] also implies

$$\frac{\partial}{\partial z}(\mathbf{V}\cdot\hat{z}) = \frac{1}{2}z\nabla^2(\mathbf{V}\cdot\hat{z}) + z_0G$$

and exploiting the simplified form of [3.16] for a symmetric function, all the required quantities in [8.2] can be calculated from knowledge of

$$V_z = (\mathbf{V}\cdot\hat{z})_{z=z_0} = \frac{3}{\pi}\left[\frac{1}{z_0}\tan^{-1} z_0 + \frac{1}{z_0^2+1} - \frac{4}{3(z_0^2+1)^2}\right]$$

(Davis *et al.*, 1981, equation C6) and

$$G(0, z_0; z_0) = \frac{2}{\pi}\left\{\frac{1}{(z_0^2+1)^2} + \int_1^\infty \frac{(z_0^2-s^2)^2}{(z_0^2+s^2)^4}\,ds\right\} = \frac{8}{\pi}\int_1^\infty\left(\frac{s}{z_0^2+s^2}\right)^4 ds$$

Hence

$$\frac{\partial V_z}{\partial z_0} = -\frac{3}{2\pi z_0}\left[\frac{1}{z_0}\tan^{-1} z_0 + \frac{1}{z_0^2+1} - \frac{2}{(z_0^2+1)^2} - \frac{16 z_0^2}{3(z_0^2+1)^3}\right]$$

$$v_z - \frac{1}{2}\nabla^2 V_z = \frac{4}{\pi z_0^2}\left[\frac{1}{z_0}\tan^{-1} z_0 + \frac{1}{z_0^2+1} - \frac{2}{(z_0^2+1)^2} - \frac{10 z_0^2}{3(z_0^2+1)^3}\right].$$

The numerical collocation technique has been successfully applied to this axisymmetric problem by Dagan, Weinbaum & Pfeffer (1982) and to its disk counterpart by Dagan, Pfeffer & Weinbaum (1982). The first mentioned authors constructed, for each half space, stream functions in terms of the unknown velocity profile at the orifice, matched them analytically to secure continuity of the kinematic and dynamic fields and then used the collocation technique described in detail by Ganatos, Pfeffer & Weinbaum (1978) to satisfy the non-slip boundary condition at the surface of the sphere. Comparison of the two methods can now be made by setting $U = 0$ (sphere moving in quiescent fluid) or $V^* = 0$ (pressure driven flow past a fixed sphere) in [8.2] in order to obtain force coefficients corresponding to those displayed in tables 6 and 10 respectively of Dagan *et al.* (1982). Agreement is remarkably good, except near the top right corner of table 3.

Table 3. Values of the force coefficients $F_z/6\pi\mu V^*c$ (quiescent fluid) and $F_z/6\pi\mu Uc$ (fixed sphere) given by formula [8.2] for various sphere radii c and sphere-to-orifice spacings z_0/c

z_0/c	$c = 0.1$	$c = 0.25$	$c = 0.5$	$c = 0.75$	$c = 1.0$	$c = 2.5$	$c = 5.0$
1.1	1.0516	1.1581	1.4593	1.8755	2.2659	2.8368	2.8380
	1.0321	1.0264	0.88548	0.82319	1.0681	1.2348	0.42048
1.25	1.0519	1.1626	1.4728	1.8675	2.2077	2.7587	2.8302
	1.0288	1.0091	0.86220	0.84774	1.0248	0.76717	0.24261
1.5	1.0526	1.1702	1.4872	1.8315	2.0863	2.4682	2.5059
	1.0225	0.97746	0.81957	0.79972	0.82430	0.36395	0.10522
2	1.0542	1.1849	1.4843	1.7132	1.8396	1.9856	1.9981
	1.0068	0.90589	0.70094	0.57509	0.45763	0.11621	0.030870
3	1.0582	1.2030	1.4080	1.4920	1.5238	1.5517	1.5537
	0.96434	0.74631	0.43752	0.26310	0.16785	0.030874	0.0078586
4	1.0626	1.2028	1.3247	1.3573	1.3676	1.3757	1.3763
	0.91014	0.59098	0.26860	0.14027	0.083788	0.014294	0.0036043
5	1.0667	1.1909	1.2618	1.2762	1.2804	1.2835	1.2837
	0.84824	0.46045	0.17554	0.086043	0.050131	0.0083269	0.0020924
6	1.0701	1.1745	1.2166	1.2238	1.2257	1.2272	1.2273
	0.78250	0.35951	0.12219	0.057996	0.033391	0.0054771	0.0013740
8	1.0735	1.1424	1.1590	1.1613	1.1619	1.1623	1.1623
	0.65181	0.22793	0.068195	0.031411	0.017892	0.0029016	0.0007268
10	1.0730	1.1173	1.1248	1.1257	1.1259	1.1261	1.1261
	0.53485	0.15383	0.043220	0.019647	0.011139	0.0017975	0.0004499

In contrast to the examples cited in the previous three sections, no simplifications of [3.19]–[3.21] are available for the off-axis sphere. **W** is given by [8.1] and the reflected velocities **U**, **V** and hence **Ω** by the results of the next section. A practical aspect of this asymmetric problem has been provided by Dagan, Pfeffer & Weinbaum (1983) who, by experimental verification and an approximate theory based on reasonable estimates of the force and torque coefficients, showed that the trajectory of a neutrally buoyant sphere departs significantly from the undisturbed fluid streamline in the vicinity of the orifice wall as the opening is approached and that in multi-particle flow into the pore, the particles tend to aggregate near the orifice wall. The pressure driven flow [8.1] provides an excellent model of the basic fluid flow near the pore.

9. ASYMMETRIC FLOWS DUE TO A STOKESLET PLACED NEAR A HOLE IN A PLANE WALL

The reflected velocity fields $-\mathbf{U}(\rho, \omega, z; \rho_0, z_0)$ and $-\mathbf{V}(\rho, \omega, z; \rho_0, z_0)$ satisfy [2.1], vanish at infinity, remain bounded as $(\rho - 1)^2 + z^2 \to 0$ and on the wall take the values prescribed by [3.14], i.e. with $r^2 = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\omega + z_0^2$,

$$\mathbf{V} = \frac{\hat{z}}{r} - \frac{z_0}{r^3}[\rho\hat{\rho} - \rho_0(\hat{\rho}\cos\omega - \hat{\omega}\sin\omega) - z_0\hat{z}]$$

$$= \left(1 - z_0\frac{\partial}{\partial z_0}\right)\frac{\hat{z}}{r} + \frac{\partial}{\partial z_0}\left[\frac{\rho\hat{\rho} - \rho_0(\hat{\rho}\cos\omega - \hat{\omega}\sin\omega)}{r}\right] \quad \text{at } z = 0,\ \rho > 1 \qquad [9.1a]$$

$$\mathbf{U} = \frac{\hat{\rho}\cos\omega - \hat{\omega}\sin\omega}{r} - \frac{(\rho_0 - \rho\cos\omega)}{r^3}[\rho\hat{\rho} - \rho_0(\hat{\rho}\cos\omega - \hat{\omega}\sin\omega) - z_0\hat{z}]$$

$$= -z_0\frac{\partial}{\partial \rho_0}\left(\frac{\hat{z}}{r}\right) + \frac{\partial}{\partial \rho_0}\left[\frac{\rho\hat{\rho} - \rho_0(\hat{\rho}\cos\omega - \hat{\omega}\sin\omega)}{r}\right]$$

$$+ \frac{2}{r}(\hat{\rho}\cos\omega - \hat{\omega}\sin\omega) \quad \text{at } z = 0,\ \rho > 1 \qquad [9.1b]$$

The normal components of [9.1a, b] give rise to velocity fields whose $\hat{z}$-components are even functions of z and which have representations of the same form as in the axisymmetric case. The tangential components of [9.1a, b] excite velocity fields whose $\hat{z}$-components are odd functions of z and hence zero over the whole plane $z = 0$. Thus write

$$\mathbf{V} = \left(1 - z_0\frac{\partial}{\partial z_0}\right)(F\hat{z} - z\,\mathrm{grad}\,F) + \int_0^\infty (u^{(1)}\hat{\rho} + v^{(1)}\hat{\omega} + w^{(1)}\hat{z}\,\mathrm{sgn}\,z)\,e^{-k|z|}\,dk \quad [9.2a]$$

$$\mathbf{U} = -z_0\frac{\partial}{\partial \rho_0}(F\hat{z} - z\,\mathrm{grad}\,F) + \int_0^\infty (u^{(2)}\hat{\rho} + v^{(2)}\hat{\omega} + w^{(2)}\hat{z}\,\mathrm{sgn}\,z)\,e^{-k|z|}\,dk \qquad [9.2b]$$

where $F(\rho, \omega, z; \rho_0, z_0)$ is an even function of z which satisfies Laplace's equation.

$$\nabla^2 F = \frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \omega^2} + \frac{\partial^2 F}{\partial z^2} = 0 \qquad [9.3]$$

everywhere except at points on the wall ($z = 0, \rho \geq 1$) and the boundary conditions

$$F(\rho, \omega, 0; \rho_0, z_0) = (\rho^2 + \rho_0^2 - 2\rho\rho_0\cos\omega + z_0^2)^{-(1/2)} \quad \text{for} \quad \rho > 1 \qquad [9.4]$$

$$F = 0[(\rho^2 + z^2)^{-(1/2)}] \quad \text{as} \quad \rho^2 + z^2 \to \infty. \qquad [9.5]$$

Also the remaining velocity fields in [9.2a, b] are such that

$$\left.\begin{aligned}\int_0^\infty (u^{(1)} + iv^{(1)})\,\mathrm{d}k &= \frac{\partial}{\partial z_0}\left(\frac{\rho - \rho_0 e^{-i\omega}}{r}\right)\\ \int_0^\infty (u^{(2)} + iv^{(2)})\,\mathrm{d}k &= \frac{\partial}{\partial \rho_0}\left(\frac{\rho - \rho_0 e^{-i\omega}}{r}\right) + \frac{2}{r} e^{-i\omega}\end{aligned}\right\} \text{ when } \rho > 1 \qquad [9.6]$$

$$\int_0^\infty (u^{(j)} + iv^{(j)})k\,\mathrm{d}k = 0 \quad \text{when} \quad \rho < 1 \quad (j = 1, 2) \qquad [9.7]$$

$$\int_0^\infty w^{(j)}\,\mathrm{d}k = 0 \quad \text{for all } \rho \quad (j = 1, 2). \qquad [9.8]$$

Consider first the function F which physically is the reflected potential that arises when a unit point charge is placed at $(\rho_0, 0, z_0)$ in the presence of an earthed perfectly conducting wall ($z = 0$) which has a hole of unit radius ($\rho < 1$). It can be calculated by solving the set of mixed boundary value problems that are obtained by using the expansion

$$\frac{1}{r} = (\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\omega + z_0^2)^{-(1/2)} = \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_0^\infty e^{-\lambda z_0} J_m(\lambda\rho_0) J_m(\lambda\rho)\,\mathrm{d}\lambda \qquad [9.9]$$

valid for $z_0 > 0$. By suitable superposition, the solution for F is evidently then given by

$$F(\rho, \omega, z; \rho_0, z_0) = \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_0^\infty \int_0^\infty e^{-(k|z| + \lambda z_0)} J_m(k\rho) J_m(\lambda\rho_0) a_m(k, \lambda)\,\mathrm{d}k\,\mathrm{d}\lambda \qquad [9.10]$$

provided, for each $m \geqslant 0$,

$$\int_0^\infty a_m(k, \lambda) J_m(k\rho)\,\mathrm{d}k = J_m(\lambda\rho) \quad (\rho > 1) \qquad [9.11a]$$

in order to satisfy (9.4) after substitution of [9.9] and

$$\int_0^\infty k a_m(k, \lambda) J_m(k\rho)\,\mathrm{d}k = 0 \quad (\rho < 1) \qquad [9.11b]$$

for an even function of z. By comparison with the solution [4.2.23] of [4.2.15/16] given by Sneddon (1966), it then follows after use of the Sonine integral

$$\int_\rho^\infty \frac{J_\nu(\lambda t)\,\mathrm{d}t}{t^{\nu-1}\sqrt{(t^2 - \rho^2)}} = \sqrt{\left(\frac{\pi}{2\lambda}\right)} \frac{J_{\nu-(1/2)}(\lambda\rho)}{\rho^{\nu-(1/2)}} \qquad [9.12]$$

(Sneddon, 1966, equation 2.1.32), that the dual integral equations [9.11a, b] for each of the set of functions $\{\alpha_m(k, \lambda);\ m \geqslant 0\}$ have solution

$$a_m(k, \lambda) = \sqrt{(k\lambda)} \int_1^\infty s J_{m+(1/2)}(ks) J_{m+(1/2)}(\lambda s)\,\mathrm{d}s \quad (m \geqslant 0), \qquad [9.13]$$

which integral exists only in the generalised sense. The result

$$\int_0^\infty k^{1/2} J_{m+(1/2)}(kt) J_m(k\rho)\, dk = \sqrt{\frac{2}{\pi t}} \left(\frac{\rho}{t}\right)^m \frac{H(t-\rho)}{\sqrt{t^2-\rho^2}} \qquad [9.14]$$

(Sneddon, 1966, equation 2.1.20), where $H(x)$ denotes the Heaviside unit function, provides verification that [9.13] solves [9.11a, b] for each $m \geqslant 0$. Expression [9.10], which reduces to [8.4a] when $\rho_0 = 0$, can be written

$$F(\rho, \omega, z; \rho_0, z_0) = \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_1^\infty s f_m(\rho, |z|; s) f_m(\rho_0, z_0; s)\, ds, \qquad [9.15]$$

where

$$f_m(\rho, z; s) = \int_0^\infty \lambda^{1/2} e^{-\lambda z} J_m(\lambda\rho) J_{m+(1/2)}(\lambda s)\, d\lambda \quad (m \geqslant 0). \qquad [9.16]$$

Evidently F is a symmetric function of the field and source positions (ρ, ω, z) and $(\rho_0, 0, z_0)$. Alternative forms of F are given below by equations [10.6] and [A3].

Now consider the integrals in [9.2a, b]. If the corresponding pressures are written

$$p_v = 2\mu\left(z_0 \frac{\partial}{\partial z_0} - 1\right)\frac{\partial F}{\partial z} + \int_0^\infty p^{(1)} e^{-k|z|}\, dk$$

$$p_u = 2\mu z_0 \frac{\partial}{\partial \rho_0}\left(\frac{\partial F}{\partial z}\right) + \int_0^\infty p^{(2)} e^{-k|z|}\, dk$$

and the Fourier series [6.7] are introduced, then, as in section 6, the equations of motion [2.1] imply that for each field (suffix omitted for convenience):

$$\left.\begin{aligned} p_m &= 2\mu q_m(k) J_m(k\rho) \\ w_m &= -q_m(k)\rho J_{m+1}(k\rho) + \frac{1}{k} r_m(k) J_m(k\rho) \end{aligned}\right\} \quad (m \geqslant 0)$$

$$\left.\begin{aligned} \pm u_m + v_m &= -q_m(k)\rho J_{m+1\pm 1}(k\rho) + \frac{1}{k} s_m^{\pm}(k) J_{m\pm 1}(k\rho) \\ q_m(k) + \frac{1}{2}[s_m^{+}(k) + s_m^{-}(k)] &= r_m(k) \end{aligned}\right\} \quad (m \geqslant 1)$$

$$u_0 = -q_0(k)\rho J_2(k\rho) + \frac{1}{k} s_0(k) J_1(k\rho), \qquad s_0(k) = r_0(k).$$

The boundary condition [9.8] implies, provided $q_0(0) = 0$ which is subsequently verified, that

$$\frac{1}{k} r_m(k) = q_m'(k) + \frac{m}{k} q_m(k) \qquad (m \geqslant 0)$$

and hence in [9.2a, b]

$$\int_0^\infty w^{(j)} \operatorname{sgn} z\, e^{-k|z|}\, dk = \frac{z}{2\mu} \int_0^\infty p^{(j)} e^{-k|z|}\, dk \quad (j = 1, 2) \qquad [9.17]$$

Further substitution for $r_m(k)$ then shows that

$$\left.\begin{aligned} u_m + v_m &= k^2 \frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{q_m(k)}{k^2} J_{m+1}(k\rho)\right] + \frac{S_m(k)}{k} J_{m+1}(k\rho) \\ u_m - v_m &= -\frac{\mathrm{d}}{\mathrm{d}k}\left[q_m(k) J_{m-1}(k\rho)\right] + \frac{S_m(k)}{k} J_{m-1}(k\rho) \end{aligned}\right\} (m \geqslant 1); \quad u_0 = k\frac{\mathrm{d}}{\mathrm{d}k}\left[\frac{q_0(k)}{k} J_1(k\rho)\right] \qquad [9.18]$$

where $S_m(k) = \frac{1}{2}[s_m^+(k) - s_m^-(k)]$ $(m \geqslant 1)$.

Now, from [6.7]:

$$u + iv = u_0 + \sum_{m=1}^{\infty} [(u_m + v_m)\,\mathrm{e}^{im\omega} + (u_m - v_m)\,\mathrm{e}^{-im\omega}]$$

and, from [9.9] and the recurrence relations, it may be shown that

$$\frac{\rho - \rho_0\,\mathrm{e}^{-i\omega}}{r} = \int_0^\infty \left\{\sum_{m=0}^{\infty} J_m(\lambda\rho_0) J_{m+1}(\lambda\rho)\,\mathrm{e}^{im\omega} - \sum_{m=1}^{\infty} \times J_m(\lambda\rho_0) J_{m-1}(\lambda\rho)\,\mathrm{e}^{-im\omega}\right\} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[-\frac{\mathrm{e}^{-\lambda z_0}}{\lambda}\right] \lambda\,\mathrm{d}\lambda.$$

Hence conditions [9.6] imply

$$\left.\begin{aligned} \int_0^\infty (u_m + v_m)_{\rho>1}\,\mathrm{d}k &= -\int_0^\infty [d_m(\lambda) - e_m(\lambda)] J_{m+1}(\lambda\rho)\,\mathrm{d}\lambda \\ \int_0^\infty (u_m - v_m)_{\rho>1}\,\mathrm{d}k &= \int_0^\infty [d_m(\lambda) + e_m(\lambda)] J_{m-1}(\lambda\rho)\,\mathrm{d}\lambda \end{aligned}\right\} (m \geqslant 1) \qquad [9.19]$$

$$\int_0^\infty (u_0)_{\rho>1}\,\mathrm{d}k = -\int_0^\infty d_0(\lambda) J_1(\lambda\rho)\,\mathrm{d}\lambda$$

where

$$\begin{aligned} d_m^{(1)}(\lambda) &= \lambda z_0\,\mathrm{e}^{-\lambda z_0} J_m(\lambda\rho_0) \quad (m \geqslant 0); & e_m^{(1)}(\lambda) &= 0 \quad (m \geqslant 1) \\ d_m^{(2)}(\lambda) &= (1 - \lambda z_0)\,\mathrm{e}^{-\lambda z_0} J'_m(\lambda\rho_0) \quad (m \geqslant 0); & e_m^{(2)}(\lambda) &= \frac{2m}{\lambda\rho_0}\,\mathrm{e}^{-\lambda z_0} J_m(\lambda\rho_0) \quad (m \geqslant 1). \end{aligned} \qquad [9.20]$$

The determination of $q_0(k)$ is now straightforward because, on substitution of [9.18], conditions [9.19] and [9.7] yield the dual integral equations

$$\int_0^\infty k^{-1} q_0(k) J_1(k\rho)\,\mathrm{d}k = \int_0^\infty d_0(\lambda) J_1(\lambda\rho)\,\mathrm{d}\lambda \quad (\rho > 1)$$

$$\int_0^\infty q_0(k) J_1(k\rho)\,\mathrm{d}k = 0 \quad (\rho < 1)$$

which by comparison with [9.11a, b] have solution

$$k^{-1}q_0(k) = \int_0^\infty a_1(k,\lambda)d_0(\lambda)\,\mathrm{d}\lambda \qquad [9.21]$$

where $a_1(k,\lambda)$ is given by [9.13]. When $\rho_0 = 0$, **V** is axisymmetric and the only non-zero contribution to the integral in [9.2a] arises from $d_0^{(1)}(\lambda) = \lambda z_0 \mathrm{e}^{-\lambda z_0}$. The resulting $\hat{z}$-component of velocity is

$$z\int_0^\infty q_0^{(1)}(k)J_0(k\rho)\,\mathrm{e}^{-k|z|}\,\mathrm{d}k = zz_0\int_0^\infty\int_0^\infty k\lambda a_1(k,\lambda)\,\mathrm{e}^{-(k|z|+\lambda z_0)}J_0(k\rho)\,\mathrm{d}k\,\mathrm{d}\lambda$$

and, in agreement with the quoted results of section 8, reduces to $zz_0G(\rho, z; z_0)$, given by [8.4b], after substitution for $a_1(k,\lambda)$ and subsequent manipulation.

However, for each $m \geqslant 1$, the integrations by parts necessitated by the substitution of expressions [9.18] into conditions [9.19] and [9.7] yield a connected pair of dual integral equations for $S_m(k)$ and $q_m(k)$, namely

$$\int_0^\infty [S_m(k) - 3q_m(k)]J_{m+1}(k\rho)\,\mathrm{d}k = 0 \qquad (\rho < 1)$$

$$\int_0^\infty k^{-1}[S_m(k) - 2q_m(k)]J_{m+1}(k\rho)\,\mathrm{d}k = -\int_0^\infty [d_m(\lambda) - e_m(\lambda)]J_{m+1}(\lambda\rho)\,\mathrm{d}\lambda \qquad (\rho > 1)$$

and

$$\int_0^\infty [S_m(k) + q_m(k)]J_{m-1}(k\rho)\,\mathrm{d}k = 0 \qquad (\rho < 1)$$

$$\int_0^\infty k^{-1}S_m(k)J_{m-1}(k\rho)\,\mathrm{d}k = \int_0^\infty [d_m(\lambda) + e_m(\lambda)]J_{m-1}(\lambda\rho)\,\mathrm{d}\lambda \qquad (\rho > 1).$$

If these are regarded as disjoint dual integral equations for $S_m(k) - 3q_m(k)$ and $S_m(k) + q_m(k)$ respectively, then by comparison with [9.11a, b], it follows that

$$S_m(k) - 3q_m(k) = -k\int_0^\infty a_{m+1}(k,\lambda)[d_m(\lambda) + \lambda^{-1}q_m(\lambda) - e_m(\lambda)]\,\mathrm{d}\lambda$$

$$S_m(k) + q_m(k) = k\int_0^\infty a_{m-1}(k,\lambda)[d_m(\lambda) + \lambda^{-1}q_m(\lambda) + e_m(\lambda)]\,\mathrm{d}\lambda \qquad [9.22]$$

Thus, by subtraction, the following Fredholm integral equation for $k^{-1}q_m(k)$ is obtained

$$4k^{-1}q_m(k) = \int_0^\infty [a_{m-1}(k,\lambda) + a_{m+1}(k,\lambda)][d_m(\lambda) + \lambda^{-1}q_m(\lambda)]\,\mathrm{d}\lambda$$
$$+\int_0^\infty [a_{m-1}(k,\lambda) - a_{m+1}(k,\lambda)]e_m(\lambda)\,\mathrm{d}\lambda \qquad [9.23]$$

Now, by suitable differentiation of the identity

$$\int_0^\infty J_\nu(kt)J_{\nu-1}(\lambda t)\,\mathrm{d}t = \frac{\lambda^{\nu-1}}{k^\nu}H(k-\lambda),$$

(Gradshteyn and Ryzhik, 1980, section 6.512), it may be shown that

$$\int_0^\infty tJ_\nu(kt)J_\nu(\lambda t)\,\mathrm{d}t = \frac{1}{k}\delta(k-\lambda) \qquad [9.24a]$$

$$\int_0^\infty tJ_{\nu+1}(kt)J_{\nu-1}(\lambda t)\,\mathrm{d}t = 2\nu\frac{\lambda^{\nu-1}}{k^{\nu+1}}H(k-\lambda) - \frac{1}{k}\delta(k-\lambda) \qquad [9.24b]$$

where $\delta(x)$ denotes the Dirac delta-function. The results [9.24a, b] respectively are sufficient to deduce that

$$\int_0^\infty a_m(k,\mu)a_m(\mu,\lambda)\,\mathrm{d}\mu = a_m(k,\lambda) \qquad (m \geqslant 0) \qquad [9.25a]$$

$$\int_0^\infty a_{m+1}(k,\mu)a_{m-1}(\mu,\lambda)\,\mathrm{d}\mu = a_{m-1}(k,\lambda) \qquad (m \geqslant 1). \qquad [9.25b]$$

Then it follows simply that the solution of [9.23] is

$$k^{-1}q_m(k) = \frac{1}{3}\int_0^t [2a_{m-1}(k,\lambda) + a_{m+1}(k,\lambda)]d_m(\lambda)\,\mathrm{d}\lambda + \frac{1}{3}\int_0^t [a_{m-1}(k,\lambda) - a_{m+1}(k,\lambda)]e_m(\lambda)\,\mathrm{d}\lambda. \qquad [9.26a]$$

By substitution in [9.22], the other mth function required below is given by

$$k^{-1}[S_m(k) - q_m(k)] = \frac{2}{3}\int_0^t [a_{m-1}(k,\lambda) - a_{m+1}(k,\lambda)]d_m(\lambda)\,\mathrm{d}\lambda + \frac{1}{3}\int_0^t [a_{m+1}(k,\lambda) + 2a_{m+1}(k,\lambda)]e_m(\lambda)\,\mathrm{d}\lambda \qquad [9.26b]$$

The end result of the above calculations is that the integrals in [9.2a, b] have the Fourier series expansions

$$\int_0^\infty (u\hat{\rho} + v\hat{\omega} + w\hat{z}\,\mathrm{sgn}\,z)\,\mathrm{e}^{-k|z|}\,\mathrm{d}k$$

$$= \hat{\rho}\sum_{m=0}^\infty \epsilon_m \cos m\omega \int_0^\infty \left[\frac{q_m(k)}{k}J'_m(k\rho)(1-k|z|) + \frac{S_m(k)-q_m(k)}{k}\cdot\frac{m}{k\rho}J_m(k\rho)\right]\mathrm{e}^{-k|z|}\,\mathrm{d}k$$

$$- 2\hat{\omega}\sum_{m=1}^\infty \sin m\omega \int_0^\infty \left[\frac{q_m(k)}{k}\cdot\frac{m}{k\rho}J_m(k\rho)(1-k|z|) + \frac{S_m(k)-q_m(k)}{k}J'_m(k\rho)\right]\mathrm{e}^{-k|z|}\,\mathrm{d}k$$

$$+ z\hat{z}\sum_{m=0}^\infty \epsilon_m \cos m\omega \int_0^\infty q_m(k)J_m(k)\,\mathrm{e}^{-k|z|}\,\mathrm{d}k \qquad [9.27]$$

where $\{q_m(k); m \geqslant 0\}$ and $\{S_m(k) - q_m(k); m \geqslant 1\}$ are given by [9.21] and [9.26a, b], $\{a_m(k,\lambda); m \geqslant 0\}$ by [9.13] and the functions $\{d_m(\lambda); m \geqslant 0\}$, $\{e_m(\lambda); m \geqslant 1\}$ appropriate to the two velocity fields by [9.20].

With the aid of [9.27], an inspection of the $\hat{z}$-component of [9.2a] and the $\hat{\rho}$-component of [9.2b] makes it evident that V_z and U_ρ are, as in section 6, symmetric functions of the field and stokeslet positions (ρ, ω, z) and $(\rho_0, 0, z_0)$. The reflected velocity $\mathbf{\Omega}$ is determined by substitution of [9.2a, b] into [3.15].

10. MODIFICATION OF SECTION 9 FOR A DISK

If instead of the plane with an orifice, S is the complementary boundary, namely the disk, then conditions [9.1a, b] apply at $\rho < 1$ and the solution forms [9.2a, b] remain valid. The function E, appearing instead of F, is given, as in [9.10], by

$$E(\rho, \omega, z; \rho_0, z_0) = \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_0^\infty \int_0^\infty e^{-(k|z| + \lambda z_0)} J_m(k\rho) J_m(\lambda\rho_0) b_m(k, \lambda)\, dk\, d\lambda \quad [10.1]$$

provided, for each $m \geqslant 0$,

$$\begin{aligned} &\int_0^\infty b_m(k, \lambda) J_m(k\rho)\, dk = J_m(\lambda\rho) \qquad (\rho < 1) \\ &\int_0^\infty k b_m(k, \lambda) J_m(k\rho)\, dk = 0 \qquad (\rho > 1). \end{aligned} \quad [10.2]$$

With the aid of the other Sonine integral

$$\int_0^\rho J_\nu(\lambda t) \frac{t^{\nu+1}\, dt}{\sqrt{\rho^2 - t^2}} = \sqrt{\frac{\pi}{2\lambda}}\, \rho^{\nu + (1/2)} J_{\nu + (1/2)}(\lambda\rho) \quad [10.3]$$

(Sneddon, 1966, equation 2.1.29), the dual integral equations [10.2] have solution

$$b_m(k, \lambda) = \sqrt{k\lambda} \int_0^1 s J_{m-(1/2)}(ks) J_{m-(1/2)}(\lambda s)\, ds \quad (m \geqslant 0) \quad [10.4]$$

verified again by [9.14]. Then, like [9.15] and [9.16], [10.1] can be written

$$E(\rho, \omega, z; \rho_0, z_0) = \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_0^1 s e_m(\rho, |z|; s) e_m(\rho_0, z_0; s)\, ds,$$

where

$$e_m(\rho, z; s) = \int_0^\infty \lambda^{1/2} e^{-\lambda z} J_m(\lambda\rho) J_{m-(1/2)}(\lambda s)\, d\lambda \qquad (m \geqslant 0) \quad [10.5]$$

Estimates of E near $z = 0$ for $\rho > 1$ can be obtained by using [10.3] to write [10.5] in the form

$$e_m(\rho, z; s) = \frac{2}{\pi} \int_0^\infty \lambda^{1/2} I_{m-(1/2)}(\lambda s) K_m(\lambda\rho) \cos \lambda z\, d\lambda,$$

valid for $z > 0$, $\rho > 1$ and $0 < s < 1$. By similarly writing [9.16] in the form

$$f_m(\rho, z; s) = \frac{2}{\pi} \int_0^\infty \lambda^{1/2} I_{m+(1/2)}(\lambda s) K_m(\lambda\rho) \sin \lambda z\, d\lambda,$$

valid for $z > 0$, $\rho > 1$ and $0 < s < 1$, estimates of F near the plane can be obtained after using [9.24a] and the summation [9.9] to rewrite [9.15] for $z > 0$ as

$$F(\rho, \omega, z; \rho_0, z_0) = [\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\omega + (z + z_0)^2]^{-(1/2)}$$

$$- \sum_{m=0}^{x} \epsilon_m \cos m\omega \int_0^1 s f_m(\rho, |z|; s) f_m(\rho_0, z_0; s)\, ds \qquad [10.6]$$

In the second part of the solution, the only change necessitated by the interchange of the ranges of validity of conditions [9.6] and [9.7] as far as [9.25a] inclusive is the replacement of $a_m(k, \lambda)$ by $b_m(k, \lambda)$, given by [10.4], for each $m \geqslant 0$. However, since [9.25b] is replaced by

$$\int_0^c b_{m-1}(k, \mu) b_{m+1}(\mu, \lambda)\, d\mu = b_{m+1}(k, \lambda) \qquad (m \geqslant 1),$$

the net effect of the changes on the solutions [9.26a, b] is that, for each $m \geqslant 1$, $a_{m-1}(k, \lambda)$ is replaced by $b_{m+1}(k, \lambda)$, $a_{m+1}(k, \lambda)$ by $b_{m-1}(k, \lambda)$, $S_m(k) - q_m(k)$ by $q_m(k) - S_m(k)$ and $e_m(\lambda)$ by $-e_m(\lambda)$ whilst $d_m(\lambda)$ is left unchanged. Thus

$$k^{-1} q_0(k) = \int_0^c b_1(k, \lambda) d_0(\lambda)\, d\lambda$$

$$k^{-1} q_m(k) = \frac{1}{3}\int_0^c [b_{m-1}(k, \lambda) + 2b_{m+1}(k, \lambda)] d_m(\lambda)\, d\lambda$$

$$+ \frac{1}{3}\int_0^c [b_{m-1}(k, \lambda) - b_{m+1}(k, \lambda)] e_m(\lambda)\, d\lambda$$

$$k^{-1}[S_m(k) - q_m(k)] = \frac{2}{3}\int_0^c [b_{m-1}(k, \lambda) - b_{m+1}(k, \lambda)] d_m(\lambda)\, d\lambda$$

$$+ \frac{1}{3}\int_0^c [2b_{m-1}(k, \lambda) + b_{m+1}(k, \lambda)] e_m(\lambda)\, d\lambda \qquad (m \geqslant 1)$$

are the results required for substitution in [9.27] and the symmetry of V_z and U_ρ with respect to (ρ, ω, z) and $(\rho_0, 0, z_0)$ is again apparent.

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APPENDIX

An alternative form of $F(\rho, \omega, z; \rho_0, z_0)$ can be readily constructed in terms of toroidal coordinates, the natural system for the given geometry, and consists of a sum of separated solutions of [9.3] which involve the Mehler conal functions $K_\tau^m = P_{-\frac{1}{2}+i\tau}^{m}$. Thus, on writing

$$\rho = \frac{\sinh \xi}{\cosh \xi - \cos \eta} \qquad z = \frac{\sin \eta}{\cosh \xi - \cos \eta},$$

with ρ_0, z_0 defined similarly in terms of ξ_0, η_0, it follows that

$$\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos \omega + (z - z_0)^2 = \frac{2\,[\cosh \xi \cosh \xi_0 - \sinh \xi \sinh \xi_0 \cos \omega - \cos(\eta - \eta_0)]}{(\cosh \xi - \cos \eta)(\cosh \xi_0 - \cos \eta_0)} \qquad \text{[A1]}$$

Now, according to Zhurina & Karmazina (1966, equation 5.8), the inverse of the distance r between the singularity $(\rho_0, 0, z_0)$ and a point $(\rho, \omega, 0)$ $(\rho > 1$, i.e. $\eta = 0$ or $2\pi)$ on the wall, is given by

$$\frac{1}{r} = (\rho^2 + \rho_0^2 - 2\rho\rho_0 \cos\omega + z_0^2)^{-(1/2)} = (\cosh\xi - 1)^{1/2}(\cosh\xi_0 - \cos\eta_0)^{1/2}$$

$$\times \int_0^\infty \frac{\cosh s(\pi - \eta_0)}{\cosh s\pi}\, ds \sum_{m=0}^{\infty} \epsilon_m(-1)^m K_s^m(\cosh\xi) K_s^{-m}(\cosh\xi_0)\cos m\omega \qquad \text{[A2]}$$

where

$$\frac{K_s^{-m}(\cosh\xi)}{K_s^m(\cosh\xi)} = \frac{\Gamma(\frac{1}{2} - m - is)}{\Gamma(\frac{1}{2} + m - is)} = \frac{(-1)^m}{\prod_{r=1}^{m}\left[s^2 + \left(r - \frac{1}{2}\right)^2\right]} \qquad (m \geq 1).$$

The formula [A1] enables the corresponding expansion for the inverse of the distance between $(\rho_0, 0, z_0)$ and (ρ, ω, z) to be readily written down, if required. Further, since $(\cosh\xi - \cos\eta)^{1/2} e^{\pm m} K_s^m(\cosh\xi)\cos m\omega$ satisfies [9.3], it follows that $F(\rho, \omega, z; \rho_0, z_0)$ is given by

$$F = (\cosh\xi - \cos\eta)^{1/2}(\cosh\xi_0 - \cos\eta_0)^{1/2} \sum_{m=0}^{\infty} \epsilon_m \cos m\omega \int_0^\infty \frac{\cosh s(\pi - \eta)\cosh s(\pi - \eta_0)}{\cosh^2 s\pi}$$

$$\times \frac{K_s^m(\cosh\xi)K_s^m(\cosh\xi_0)}{\prod_{r=1}^{m}\left[s^2 + \left(r - \frac{1}{2}\right)^2\right]} \qquad \text{[A3]}$$

with the symmetrical dependence on the field and source positions again apparent. In the axisymmetric case $\rho_0 = 0$, i.e. $\xi_0 = 0$, this expression reduces, because $K_s(1) = 1$ and $K_s^m(1) = 0$ $(m \geq 1)$, to the form given by equations [3.1], [3.2] of Davis *et al.* (1981). Unfortunately the practical use of the formula [A3] for F is likely to be restricted by the limited knowledge of the values of the conal functions.

For the complementary problem involving the disk, the toroidal coordinate lies in the range $-\pi$ to π and the corresponding form of [A3] is obtained by replacing $\pi - \eta$, $\pi - \eta_0$ by η, η_0 respectively.